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Explicit formulae for the valuation of European options with price impacts

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Abstract

In this work, we analyze the effect of trading a *large* position of vanilla European options where the underlying price S follows a multi-period binomial model. Due to the large size of the transaction, we expect that not only the price of the derivative but also the price of the underlying S, should be subject to price impacts. As a byproduct, the valuation of derivatives should be analyzed taking into account the latter effects. In order to do so, besides assuming that the price process S can be modeled using a multi-period binomial model, we also assume that the trading impacts affect the price S in a multiplicative way. Furthermore, our analysis is carried out in discrete time to better trace the effects of price impacts, and conclude for instance, that the strike price should be itself a function of the size of the trade, and the parameterized market impacts. We provide explicit formulae for the price of European options under market impacts as well as numerical examples to illustrate our results. Code in the statistical package R can be provided upon request.

Keywords: Price impacts, valuation of derivatives, multi-period binomial model.

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1 Introduction

Unlike what is usually assumed about a small agent in many economic models that takes prices as given, this work deals with the opposite idea: a *big* agent (trader, investor, underwriter, buyer, or vendee) that can affect the price when he trades a large number of units of securities. This leads to the concept of *price impact*, which is a common and well-analyzed topic in several market microstructure models.

We briefly explain the price impact concept by means of a Limit Order Book (LOB). Namely, a LOB is a data record of orders to sell and buy assets that have not been matched with an order of immediate execution. The former type is called a limit order, whereas the latter type is known as a market order. The LOB contains, among other information, a list of prices of certain securities and the volumes offered at those prices. The volume is an important point in our analysis. If the investor executes a small market sale (purchase) order of a security when there is a large volume of limit orders to buy (sell), it is very likely for the market order to be accomplished with the volume of limit orders at the highest (lowest) buying (selling) price. While for an order with a very large volume, it is possible for the agent to descend (escalate) the price levels until his order is fully achieved.

In general, there are at least two kinds of impacts in which the investor can modify the price: *temporary and permanent impact*. Temporary impact refers to a short-term imbalance between demand and supply caused by our trading. Such trades lead to temporary price deviations away from equilibrium. On the other hand, permanent impact refers to a change in the equilibrium price caused by trading. The latter reveals a change in the perception of the other agents over long-term expectations.

This work is focused on the outstanding effects that these two impacts have on the valuation of derivatives in discrete time. It is a known fact that under mild assumptions, a financial derivative can be priced using a replicating portfolio. In turn, financial institutions use these portfolios to hedge their risk. The analysis of the price impacts on the underlying assets should also be useful to analyze changes in the price of the derivatives, caused by the purchase of the so-called replicating portfolio. In this work, we analyze the effect of trading a large position M of financial derivatives where the underlying price S follows a multi-period binomial model. Under this scenario, we will show that if an investor hedges a large position of vanilla European options with the replicating portfolio, then both the price of the underlying asset S, and the price of the derivative will be affected due to market impact.

Regarding the concept of price impact, we refer to Bertsimas and Lo [4], as well as to the work of Almgren and Chriss [1, 2]. With respect to the literature on the pricing and hedging of derivatives with market impact and its link with optimal execution, see for instance Almgren and Li [3], Cetin, Jarrow and Protter [6] or Guéant and Pu [8]. In particular, in Gökay and Soner [7] the authors propose an algorithm to calculate the option's price. Alternatively in this work, we derive explicit formulae to compute the price of European options under the assumption of multiplicative price impacts. An interesting consequence of our methodology is that in the case of call and put options, the strike price becomes itself a deterministic process which in turn is a function of the size of the trade.

The rest of the paper is organized as follows. In the next section, we describe our price model with both impacts (temporary and permanent) indicating how these affect the price. Section 3 is devoted to the study of price impacts in the case of options (call and put). In particular, we show that the hedging of a great number of derivatives may lead to a mispricing of the contract if market impact is not considered, we provide some numerical examples. We conclude this work with some final remarks in Section 4.

2 The price impact model

We shall denote by $\alpha_t \in \mathbb{R}$ the amount of shares traded at time t. In the case in which α_t is positive, we will say that the trader has executed a purchase of shares. Otherwise, if α_t is negative, we will say that the trader has executed a sale. Also, at time t, we denote by S_t the price process in the market, while $\hat{S}_t$ denotes the actual price at which the investor is trading.

We assume that there is some price impact, which affects the price proportionally to its level. Namely, if α_t units of asset S are traded at an observed initial price of, say, $S_t = 100$, and if we assume that this trading affects the price leading it to $\hat{S}_t = 105$, then any other initial price that is higher, say $S_t = 1000$, will increase to the level of $\hat{S}_t = 1050$ due also to the impact. Under this assumption, the return $\frac{\hat{S}_t - S_t}{S_t}$ is assumed to be proportional to its logarithmic approximation, i.e., $\log\left(\frac{\hat{S}_t}{S_t}\right)$. Thus, the price follows the dynamic

$$\log\left(\frac{\hat{S}_t}{S_t}\right) = \gamma\alpha_t \quad \text{for } t = 0, 1, \ldots,$$

where $\gamma > 0$ denotes an arbitrary impact, and consequently

$$\hat{S}_t = S_t e^{\gamma\alpha_t} \quad \text{for } t = 0, 1, \ldots. \tag{2.1}$$

It is worth mentioning that this type of function, which is related to price impact, has been previously considered for continuous-time models by Cetin et. al. [6]. In particular, the authors put forward the so-called *supply curve*, where the traders, not being price takers anymore, pay an amount depending on the quantity they are trading.

Let us now distinguish between the two types of impact we shall be working within the remainder.

Temporary price impact: The first impact refers to the case when the trading only affects the instant when it is executed and its influence in the future is negligible. This impact is called *temporary price impact* (or simply *temporary impact*) and will be parameterized by λ ($\gamma = \lambda$ in this case). For example, if we consider a trade of α_0 shares at time $t = 0$ and there is no trade at $t = 1$, then we have the following:

$$\begin{aligned} \hat{S}_0 &= S_0 e^{\lambda\alpha_0} = s e^{\lambda\alpha_0} \qquad \text{where } S_0 = s, \\ \hat{S}_1 &= S_1. \end{aligned}$$

We see that the impact of the first period is not considered for the next periods.

Permanent price impact: When more than one trade is executed in a period of time N, part of the former trades still affect the subsequent ones. For instance, agents that observe a large volume order may reconsider their expectations over the price of the security. We can model this fact by letting the previous impacts prevail during the whole sequence of times $\mathcal{T} = \{0, 1, \ldots N\}$. We refer to this impact as *permanent price impact* (or *permanent impact*) and will be parameterized by β ($\gamma = \beta$ in this case). For example, if two trades were made at times $t = 0$ and $t = 1$, then the price is affected as follows:

$$\begin{aligned} \hat{S}_0 &= s e^{\beta\alpha_0}, \\ \hat{S}_1 &= \left(S_1 e^{\beta\alpha_0}\right) e^{\beta\alpha_1} = S_1 e^{\beta\alpha_0 + \beta\alpha_1}. \end{aligned}$$

We note that the price $S_1 e^{\beta\alpha_0}$ is the one observed before the trade at $t = 1$ is done. Furthermore, if in the previous example we analyze the price at $t = 1$, given that no other trades are made at this time, we observe the following price

$$\hat{S}_1 = S_1 e^{\beta\alpha_0}.$$

Although it is named *permanent*, for the purposes of this paper it is sufficient that the impact remains for the entire interval of time for which the trading is allowed.

In summary, by considering the two price impacts, the price at the $t-$th period becomes

$$\hat{S}_t = S_t e^{\lambda\alpha_t + \beta\sum_{j=0}^{t}\alpha_j}, \quad t = 0, \cdots, N.$$

3 European option valuation, the binomial model

We first analyze the pricing of vanilla European options when the price impacts are present, and given that a large trade of derivatives $M >> 0$ has been carried out in the one-period case. For a brief summary of the pricing of a derivative in the case in which the price process is Bernoulli, and there is no price impact, see Appendix A.

3.1 One period case.

The dynamics with price impact. Consider a trader that sells $M >> 0$ contracts of a financial derivative of an underlying whose price is S, in a market with price impacts. The price S is a stochastic process given by

$$S_0 = s, \qquad S_1 = \begin{cases} s \cdot u & \text{with probability } p \\ s \cdot d & \text{with probability } 1-p \end{cases} \tag{3.1}$$

with $0 < d < u$. The underwriter of the contract will determine the fair price of the instrument constructing a replicating portfolio that contains bonds and the underlying security. In particular, we will assume that x represents the number of bonds with return rate R and normalized price \$1. In addition, let α_0 be the units of the underlying stock traded at $t = 0$.

Let $h = (x, \alpha_0)$ be the replicating portfolio of the derivative. Considering market impacts, the value of the portfolio at $t = 0$ equals

$$\begin{aligned} V_0^h &= x + \alpha_0 \hat{S}_0 \\ &= x + \alpha_0 s e^{\lambda\alpha_0 + \beta\alpha_0}, \end{aligned}$$

where $\hat{S}_0$ corresponds *to the effective value of the asset* S taking into account the impacts on the price process, due to the transaction size α_0. Notice that we are using the same assumption on α_0 as before: $\alpha_0 > 0$ means that the stock is bought, whereas $\alpha_0 < 0$ means that the stock is sold.

Remark 3.1. *Let us assume that the derivative is a vanilla option. Notice that under the assumption of market impacts, and that the strike price K remains fixed, one can always find an ← a??number M such that, if an amount M of derivatives are traded, then the option will always be executed. For example, consider a call option in the one-period binomial model with strike price K, and such that $sd < K < su$, i.e.*

$$\Phi(sZ) := \max\{sZe^{\beta\alpha_0} - K, 0\}.$$

It is straightforward to show that if

$$\alpha_0 > \frac{1}{\beta} \log\left(\frac{K}{sd}\right),$$

then

$$sde^{\beta\alpha_0} - K > 0,$$

and, therefore, $K < sde^{\beta\alpha_0} < sue^{\beta\alpha_0}$. This situation is similar to that of Remark A.1.2, and implies that, under price impacts, if

$$M = \alpha_0 > \frac{1}{\beta} \log\left(\frac{K}{sd}\right), \tag{3.2}$$

then, the derivative will always be executed. In order to avoid this situation, we will assume that K can be negotiated depending on the impacts and α_0.

As a consequence of the previous Remark, within our model, we will assume that both the holder and the underwriter know that there is a price impact. According to this, the strike price will be a function of this fact, namely $\hat{K} := Ke^{\beta\alpha_0}$. First, note that under the previous assumptions, the value of a call option (A.2) at expiration equals

$$\hat{\Phi}(sZ) := \Phi(sZ)e^{\beta\alpha_0} = \max\{sZe^{\beta\alpha_0} - Ke^{\beta\alpha_0}, 0\} = \max\{sZe^{\beta\alpha_0} - \hat{K}, 0\}. \tag{3.3}$$

Alternatively, in the case of a put option the contract has payoff

$$\hat{\Phi}(sZ) = \max\{Ke^{\beta\alpha_0} - sZe^{\beta\alpha_0}, 0\} = \max\{\hat{K} - sZe^{\beta\alpha_0}, 0\}. \tag{3.4}$$

In general, as long as the payoff function is linearly homogeneous, i.e

$$\Phi(\delta sZ) = \delta\Phi(sZ) \quad \forall\, \delta > 0, \tag{3.5}$$

then the inequalities in (A.1) hold since $e^{\beta\alpha_0} \geq 0$.

Next, we have that $M \cdot \hat{\Phi}$ is the value of the total amount of contracts acquired (or sold). Thus, at time $t = 1$, the following identities should hold

$$V_1^h = \begin{cases} x(1+R) + \alpha_0 s e^{\beta\alpha_0} u = M\hat{\Phi}(su) \\ x(1+R) + \alpha_0 s e^{\beta\alpha_0} d = M\hat{\Phi}(sd). \end{cases} \tag{3.6}$$

Furthermore, as we assume that $d < u$, now we solve α_0 and x from (3.6), to get

$$\alpha_0 = \frac{M}{se^{\beta\alpha_0}} \frac{\hat{\Phi}(su) - \hat{\Phi}(sd)}{u-d} = \frac{M}{se^{\beta\alpha_0}} \frac{(\Phi(su) - \Phi(sd))e^{\beta\alpha_0}}{u-d} = \frac{M}{s} \frac{\Phi(su) - \Phi(sd)}{u-d}, \tag{3.7}$$

and

$$x = \frac{M}{(1+R)} \frac{u\hat{\Phi}(sd) - d\hat{\Phi}(su)}{u-d} = \frac{M}{(1+R)} \frac{(u\Phi(sd) - d\Phi(su))e^{\beta\alpha_0}}{u-d}. \tag{3.8}$$

From (3.7), we have that:

$$\begin{aligned} &\text{Put option} \;\Rightarrow\quad \alpha_0 < 0 \;\;\Rightarrow\; \text{underwriter is selling the stock.} \\ &\text{Call option} \;\Rightarrow\quad \alpha_0 > 0 \;\;\Rightarrow\; \text{underwriter is buying the stock.} \end{aligned} \tag{3.9}$$

Thus, we conclude that the value of the contract at time $t = 0$ should be:

$$\begin{aligned} V_0^h &= \frac{M}{1+R}\left(\frac{(u\Phi(sd) - d\Phi(su))e^{\beta\alpha_0}}{u-d}\right) + \frac{Mse^{\lambda\alpha_0+\beta\alpha_0}}{s}\left(\frac{\Phi(su) - \Phi(sd)}{u-d}\right) \\ &= \frac{M}{1+R}\left(\frac{u\hat{\Phi}(sd) - d\hat{\Phi}(su)}{u-d}\right) + \frac{M(1+R)e^{\lambda\alpha_0}}{1+R}\left(\frac{\hat{\Phi}(su) - \hat{\Phi}(sd)}{u-d}\right) \\ &= \frac{M}{1+R}\left[\left\{\frac{(1+R)e^{\lambda\alpha_0} - d}{u-d}\right\}\hat{\Phi}(su) + \left\{\frac{u - (1+R)e^{\lambda\alpha_0}}{u-d}\right\}\hat{\Phi}(sd)\right]. \end{aligned} \tag{3.10}$$

Remark 3.2.

- *Notice that the measure $\mathbb{Q}$ generated by this process*

$$q_u = \frac{(1+R)e^{\lambda\alpha_0} - d}{u-d}, \quad q_d = \frac{u - (1+R)e^{\lambda\alpha_0}}{u-d}$$

yields a sub (resp. super) martingale measure when $\alpha_0 > 0$ (resp. $\alpha_0 < 0$) since the discounted price process S_1, under the new measure $\mathbb{Q}$ equals,

$$\frac{1}{(1+R)}\mathbb{E}^{\mathbb{Q}}[S_1|S_0] = se^{\lambda\alpha_0} \begin{cases} \geq s = S_0 & \text{if } \alpha_0 > 0 \\ \leq s = S_0 & \text{if } \alpha_0 < 0, \end{cases}$$

in other words, the price process is drifting under $\mathbb{Q}$.

- *Observe that the permanent impact affects the valuation just through the new contract function $\hat{\Phi}$, while the temporary impact is the only one affecting the martingale measure $\mathbb{Q}$. To see this assume that $\lambda = 0$ and observe that there are no changes between* (3.10) *and the well-known expression* (A.6).

- *On the other hand, by assuming $\beta = 0$, the underwriter has to consider the price $se^{\lambda\alpha_0}$ at $t = 0$, when he trades α_0 shares. This modification on the price is not considered by the holder in $t = 1$. This leads to an misvaluation of the derivative.*

Example 3.3. *Let $s = 10$, $u = 1.2$, $d = 0.8$, $\lambda = 0.001$, $\beta = 0$ and $R = 0$. Furthermore, assume the contract is a call option with strike price $K = 11$ and the writer of the contract sells $M = 10$ contracts. It follows that $\alpha_0 = 10/4$ (the underwriter has to buy 10/4 units of the asset), and thus the call price per contract, without assuming market impact is*

$$\bar{V}_0^h = 5.$$

In contrast, the price assuming market impact is

$$V_0^h = 5.06258.$$

Furthermore, when we assume that all the previous parameters remain the same, except for the permanent impact, which is now $\beta = 0.01$. Then $\alpha_0 = 10/4$, $\hat{K} = 11.0275$, and

$$V_0^h = 5.07525.$$

In any case ($\lambda > 0$ or $\beta > 0$), there is a variation on the call prices when a price impact is considered.

3.2 N periods case.

3.2.1 The model

The reasoning carried out in the one-period case can be applied to a multiperiod binomial model. In particular, let us denote by $\mathcal{Z} = \{Z_t : \ 1 \le t \le N\}$ the stochastic process of i.i.d. random variables defined on a (fixed) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that for all t, Z_t follows a Bernoulli distribution

$$Z_t = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p. \end{cases}$$

The process $\mathcal{Z}$ as described above has 2^N different paths. In fact, for any time $0 < t \le N$ there are 2^t different paths that we index with $j \in \{0, 1, \dots 2^t - 1\}$. For example, for $N = 3$ we have

$t = 1$	$t = 2$	$t = 3$
		$j = 0 \to (Z_1, Z_2, Z_3) = (ddd)$
	$j = 0 \to (Z_1, Z_2) = (dd)$	$j = 1 \to (Z_1, Z_2, Z_3) = (ddu)$
		$j = 2 \to (Z_1, Z_2, Z_3) = (dud)$
$j = 0 \to (Z_1) = (d)$	$j = 1 \to (Z_1, Z_2) = (du)$	$j = 3 \to (Z_1, Z_2, Z_3) = (duu)$
		$j = 4 \to (Z_1, Z_2, Z_3) = (udd)$
$j = 1 \to (Z_1) = (u)$	$j = 2 \to (Z_1, Z_2) = (ud)$	$j = 5 \to (Z_1, Z_2, Z_3) = (udu)$
		$j = 6 \to (Z_1, Z_2, Z_3) = (uud)$
	$j = 3 \to (Z_1, Z_2) = (uu)$	$j = 7 \to (Z_1, Z_2, Z_3) = (uuu)$

For each $0 < t \le N$ and $j \in \{0, \dots, 2^t - 1\}$ we define

$$r_t(j) := \begin{cases} j & \text{for } t = 1 \\ j - \sum_{k=1}^{t-1} \left\lfloor \frac{j}{2^{t-k}} \right\rfloor & \text{for } t \ge 2, \end{cases} \tag{3.11}$$

where $\lfloor \bullet \rfloor$ denotes the "floor" function, i.e. $\lfloor x \rfloor = \max\{m \in \mathbb{Z} | m \le x\}$.

Observe that $r_t(j)$ is the number of times that the process has taken the value u until step t along the path j.

Now, we are looking for a self-financing portfolio process $\mathcal{Y} = \{Y_t : 0 \leq t \leq N\}$ that replicates the value of the derivative at every node of any path j. To this end, we define an initial value of wealth Y_0, which will be determined later, and we use it to buy a portfolio made up of α_0 units of the underlying asset, and x_0 units of the bond, i.e.,

$$Y_0 = x_0 + \alpha_0 s e^{\lambda\alpha_0 + \beta\alpha_0} = x_0 + \alpha_0 \widehat{S}_0 e^{\lambda\alpha_0}, \tag{3.12}$$

where $\hat{S}_0 = s e^{\beta\alpha_0}$. We highlight that for this section, the notation $\hat{S}_j$, for $j = 0, \ldots, t$ is used for the price with permanent impact only, and we write the effect of a temporary impact explicitly when it corresponds.

For $t = 1$, we have $j \in \{0, 2^1 - 1\} = \{0, 1\}$, $r_1(0) = 0$ and $r_1(1) = 1$. Thus the wealth Y_1 is described as follows

$$\begin{aligned} Y_1(1) &= x_0(1+R) + \alpha_0 s e^{\beta\alpha_0} u \\ &= (Y_0 - \alpha_0 \widehat{S}_0 e^{\lambda\alpha_0})(1+R) + \alpha_0 \widehat{S}_0 u, \end{aligned}$$

and

$$\begin{aligned} Y_1(0) &= x_0(1+R) + \alpha_0 s e^{\beta\alpha_0} d \\ &= (Y_0 - \alpha_0 \widehat{S}_0 e^{\lambda\alpha_0})(1+R) + (\alpha_0)\hat{S}_0 d, \end{aligned}$$

where we have used the equation (3.12) to express x_0 in terms of Y_0 and α_0.

Additionally, let $\alpha_t(j)$ and $x_t(j)$ be, respectively, the amount of the underlying, and the number of bonds that the underwriter holds at stage t along path j. In general, we define recursively forward $\mathcal{Y}$ as

$$\begin{aligned} Y_{t+1}(2j+1) &= \left(Y_t(j) - \alpha_t(j)\widehat{S}_t(j)e^{\lambda\alpha_t(j)}\right)(1+R) + \alpha_t(j)\widehat{S}_t(j)u \\ Y_{t+1}(2j) &= \left(Y_t(j) - \alpha_t(j)\widehat{S}_t(j)e^{\lambda\alpha_t(j)}\right)(1+R) + \alpha_t(j)\widehat{S}_t(j)d \end{aligned} \tag{3.13}$$

where

$$\widehat{S}_t(j) = s u^{r_t(j)} d^{t-r_t(j)} e^{\beta \sum_{k=0}^{t} \alpha_k(\lfloor j/2^{t-k} \rfloor)}, \tag{3.14}$$

and $2j+1$ from $Y_{t+1}(\cdot)$ indicates that the transition from t to $t+1$ was generated by $Z_{t+1} = u$, whilst $2j$ was generated by $Z_{t+1} = d$. The equations (3.13) are known as *wealth equations*.

Let $\mathcal{V} = \{V_t; 0 \leq t \leq N\}$ be the value process of the derivative that evolves according to the $\mathcal{Z}$ process. This means that for some $\omega_j \in \Omega$, $V_t(Z_1(\omega_j), \ldots, Z_t(\omega_j)) = V_t(j)$ represents the value of the derivative at stage t along path j. Finally, as a consequence of the previous definitions, the value at time N of this process is the contract function which equals

$$\begin{aligned} V_N(j) &= M\Phi\left(s u^{r_N(j)} d^{N-r_N(j)} e^{\beta \sum_{k=0}^{N-1} \alpha_k(\lfloor j/2^{N-k} \rfloor)} \right) \\ &= M e^{\beta \sum_{k=0}^{N-1} \alpha_k(\lfloor j/2^{N-k} \rfloor)} \Phi\left(s u^{r_N(j)} d^{N-r_N(j)} \right). \end{aligned} \tag{3.15}$$

3.2.2 Main results: the Algorithm and the Valuation

Next, our task is to show that the portfolio process $\mathcal{Y}$ takes the same values as the derivative process $\mathcal{V}$ for each stage t at any path j. For this, we have the following Theorem.

Theorem 3.4. *Consider a N-claim $V_N = M\Phi(S_N)$ as in (3.15) according to the binomial process $\mathcal{Z}$. For $t \in \{0, \ldots, N\}$ and $j \in \{0, \ldots 2^t - 1\}$, let us define recursively backward in time the sequence of random variables $V_{N-1}, V_{N-2}, \ldots, V_0$ by*

$$V_t(j) = \frac{1}{1+R}\Big[q_t^1(j)V_{t+1}(2j+1) + q_t^0(j)V_{t+1}(2j)\Big], \tag{3.16}$$

where

$$q_t^1(j) := \frac{(1+R)e^{\lambda\alpha_t(j)} - d}{u-d} \quad \text{and} \quad q_t^0(j) := \frac{u - (1+R)e^{\lambda\alpha_t(j)}}{u-d}, \tag{3.17}$$

and $\alpha_t(j)$ is given by

$$\alpha_t(j) = \frac{V_{t+1}(2j+1) - V_{t+1}(2j)}{\hat{S}_t(j)(u-d)}. \tag{3.18}$$

Additionally, let us define

$$x_t(j) = \frac{V_{t+1}(2j)u - V_{t+1}(2j+1)d}{(1+R)(u-d)}. \tag{3.19}$$

If we set $Y_0 = V_0$ and define recursively forward in time the portfolio values $Y_1, Y_2, \ldots, Y_N$ by (3.13), then we will have

$$V_N(j) = Y_N(j), \qquad \forall\, j \in \{0, \cdots, 2^N - 1\}. \tag{3.20}$$

Proof. The proof is done by induction. For $t = 1$, from (3.13) we have

$$Y_1(i) = \Big(Y_0 - \alpha_0\widehat{S}_0 e^{\lambda\alpha_0}\Big)(1+R) + \alpha_0\widehat{S}_0 u^i d^{1-i}, \qquad \text{for } i = 0, 1. \tag{3.21}$$

As we set $Y_0 = V_0$, then we can rearrange the previous equation as

$$Y_1(i) = V_0(1+R) + \alpha_0\widehat{S}_0\left(u^i d^{1-i} - (1+R)e^{\lambda\alpha_0}\right), \qquad \text{for } i = 0, 1. \tag{3.22}$$

Now, we use (3.16) and (3.18) to get

$$\begin{aligned}
Y_1(i) &= V_0(1+R) + \alpha_0\widehat{S}_0\left(u^i d^{1-i} - (1+R)e^{\lambda\alpha_0}\right) \\
&= \frac{1}{1+R}\Big[q_0^1(0)V_1(1) + q_0^0(0)V_1(0)\Big](1+R) + \frac{V_1(1) - V_1(0)}{\widehat{S}_0(u-d)}\widehat{S}_0\left(u^i d^{1-i} - (1+R)e^{\lambda\alpha_0}\right) \\
&= q_0^1(0)V_1(1) + q_0^0(0)V_1(0) + \big[V_1(1) - V_1(0)\big]\left(\frac{u^i d^{1-i} - (1+R)e^{\lambda\alpha_0}}{u-d}\right)
\end{aligned}$$

Observe that

$$\frac{u^i d^{1-i} - (1+R)e^{\lambda\alpha_0}}{u-d} = \begin{cases} -q_0^1(0) & \text{for } i = 0 \\ q_0^0(0) & \text{for } i = 1 \end{cases}$$

Therefore, we have

$$\begin{aligned}
Y_1(i) &= q_0^1(0)V_1(1) + q_0^0(0)V_1(0) + \big[V_1(1) - V_1(0)\big]\left\{\begin{array}{ll} -q_0^1(0) & \text{for } i = 0 \\ q_0^0(0) & \text{for } i = 1 \end{array}\right\} \\
&= \left\{\begin{array}{ll} q_0^1(0)V_1(1) + q_0^0(0)V_1(0) - q_0^1(0)V_1(1) + q_0^1(0)V_1(0) & \text{for } i = 0 \\ q_0^1(0)V_1(1) + q_0^0(0)V_1(0) + q_0^0(0)V_1(1) - q_0^0(0)V_1(0) & \text{for } i = 1 \end{array}\right\} \\
&= \left\{\begin{array}{ll} q_0^0(0)V_1(0) + q_0^1(0)V_1(0) = V_1(0) & \text{for } i = 0 \\ q_0^1(0)V_1(1) + q_0^0(0)V_1(1) = V_1(1) & \text{for } i = 1 \end{array}\right\} = V_1(i).
\end{aligned}$$

where we have use the fact that $q_0^1(0) + q_0^0(0) = 1$ as it can be deduced from (3.17). Thus we have shown that Theorem 3.4 holds for our base of induction, $t = 1$.

For the induction step, we assume that (3.20) holds for $t < N$ and show it for $t + 1$. In addition, we will show the case of $Z_{t+1} = u$, however, the case $Z_{t+1} = d$ can be proved analogously.

Let

$$\begin{aligned} Y_{t+1}(2j+1) &= \left(Y_t(j) - \alpha_t(j)\hat{S}_t(j)e^{\lambda\alpha_t(j)}\right)(1+R) + \alpha_t(j)\hat{S}_t(j)u \\ &= Y_t(j)(1+R) + \alpha_t(j)\hat{S}_t(j)\left(u - (1+R)e^{\lambda\alpha_t(j)}\right), \end{aligned}$$

and by taking the value of $\alpha_t(j)$ given by (3.18), we get

$$Y_{t+1}(2j+1) = V_t(j)(1+R) + \frac{V_{t+1}(2j+1) - V_{t+1}(2j)}{\hat{S}_t(j)(u-d)}\hat{S}_t(j)(u - (1+R)e^{\lambda\alpha_t(j)}),$$

where we have used our induction hypothesis, $Y_t(j) = V_t(j)$. Then

$$\begin{aligned} Y_{t+1}(2j+1) &= V_t(j)(1+R) + \frac{V_{t+1}(2j+1) - V_{t+1}(2j)}{\hat{S}_t(j)(u-d)}\hat{S}_t(j)(u - (1+R)e^{\lambda\alpha_t(j)}) \\ &= V_t(j)(1+R) + \Big(V_{t+1}(2j+1) - V_{t+1}(2j)\Big)\left(\frac{u - (1+R)e^{\lambda\alpha_t(j)}}{u-d}\right) \\ &= V_t(j)(1+R) + \Big(V_{t+1}(2j+1) - V_{t+1}(2j)\Big)q_t^0(j). \end{aligned}$$

We use (3.16) to get

$$\begin{aligned} Y_{t+1}(2j+1) &= \frac{(1+R)}{1+R}\left[q_t^1(j)V_{t+1}(2j+1) + q_t^0(j)V_{t+1}(2j)\right] + q_t^0(j)V_{t+1}(2j+1) - q_t^0(j)V_{t+1}(2j) \\ &= q_t^1(j)V_{t+1}(2j+1) + q_t^0(j)V_{t+1}(2j) + q_t^0(j)V_{t+1}(2j+1) - q_t^0(j)V_{t+1}(2j) \\ &= q_t^1(j)V_{t+1}(2j+1) + q_t^0(j)V_{t+1}(2j+1) \\ &= (q_t^1(j) + q_t^0(j))V_{t+1}(2j+1) = V_{t+1}(2j+1). \end{aligned}$$

In the last equality, we have use the fact that $q_t^1(j) + q_t^0(j) = 1$ as it can be deduced from (3.17).Thus we have shown $Y_{t+1}(j) = V_{t+1}(j)$ for any $j \in \{0, \ldots, 2^{t+1} - 1\}$ and $t < N$, this includes the case $t + 1 = N$, hence we have proved (3.20). □

One of the main problems that arise when one begins to determine the values $V_t(2j + 1)$, $V_t(2j)$, $\alpha_t(j)$ and $x_t(j)$ using the formulae presented in Theorem 3.4, is that it seems to be a circular argument. On the one hand, in order to determine $\alpha_{N-1}(j)$, we need the values of $V_N(2j+1)$ and $V_N(2j)$ (equation (3.18) with $t = N - 1$), and on the other hand, these latter values, in the same way, need $\alpha_t(j)$ for $t \in \{0, \ldots, N - 1\}$ and $j \in \{0, \ldots, 2^t - 1\}$ as we see in formula (3.15). This last idea seems to call for a fixed point argument however formula (3.23), which is presented below, along with the algorithm presented next solve the problem.

3.2.3 Algorithm to obtain $V_t(j)$, $\alpha_t(j)$, $x_t(j)$ and $K(j)$.

Let us introduce the formula

$$\widehat{V}_t(j) := V_t(j)e^{-\beta\sum_{k=0}^{t-1}\alpha_k\left(\left\lfloor\frac{j}{2^{t-k}}\right\rfloor\right)} \tag{3.23}$$

which allows us to write $\alpha_{N-1}(j)$ as

$$\alpha_{N-1}(j) = \frac{\widehat{V}_N(2j+1) - \widehat{V}_N(2j)}{su^{r_{N-1}(j)}d^{N-1-r_{N-1}(j)}(u-d)} = M\frac{\Phi\left(su^{r_N(2j+1)}d^{N-r_N(2j+1)}\right) - \Phi\left(su^{r_N(2j)}d^{N-r_N(2j)}\right)}{su^{r_{N-1}(j)}d^{N-1-r_{N-1}(j)}(u-d)}. \tag{3.24}$$

which only depends on known variables and parameters. In the next step, using (3.17), for $t = N-1$, we can obtain $q_N^h(j)$, $h \in \{0,1\}$, and then we obtain $\widehat{V}_{N-1}(j)$ with the formula

$$\widehat{V}_{N-1}(j) = \frac{1}{1+R}\left[q_{N-1}^1(j)\widehat{V}_N(2j+1) + q_{N-1}^0(j)\widehat{V}_N(2j)\right]e^{\beta\alpha_{N-1}(j)} \tag{3.25}$$

The next steps consist of solving backward using

$$\alpha_t(j) = \frac{\widehat{V}_{t+1}(2j+1) - \widehat{V}_{t+1}(2j)}{su^{r_t(j)}d^{t-r_t(j)}(u-d)}, \tag{3.26}$$

then, we obtain $q_t^h(j)$, $h \in \{0,1\}$, with (3.17), and next, we solve the equation

$$\widehat{V}_t(j) = \frac{1}{1+R}\left[q_t^1(j)\widehat{V}_{t+1}(2j+1) + q_t^0(j)\widehat{V}_{t+1}(2j)\right]e^{\beta\alpha_t(j)} \tag{3.27}$$

for $t \in \{0, \ldots, N-2\}$. Finally, we can solve for $V_t(j)$ using equation (3.23) and get

$$V_t(j) = \widehat{V}_t(j)e^{\beta\sum_{k=0}^{t-1}\alpha_k\left(\left\lfloor\frac{j}{2^{t-k}}\right\rfloor\right)}. \tag{3.28}$$

The important fact about the steps described before is that $\alpha_t(j)$, $q_t^h(j)$ and $\widehat{V}_t(j)$ ($t \in \{0, \ldots, N-2\}$, $h \in \{0,1\}$ and $j\{0, \ldots, 2^t\}$) only depend on variables obtained in the previous steps, $t+1, \ldots, N$.

We verify the validity of equations (3.24), (3.25), (3.26), and (3.27) in the Appendix B. The step-by-step pseudocode is shown next.

3.2.4 Pseudocode

- Obtain values s, R and M
- Choose parameters K, N and the function $\Phi(\cdot)$
- Estimate parameters u, d, λ and β

1. For $j \in \{0, 1, \ldots, 2^N - 1\}$, define

   $$\widehat{V}_N(j) := M\Phi\left(su^{r_N(j)}d^{N-r_N(j)}\right)$$

   where $r_N(j)$ can be computed using (3.11) with $t = N$

   End

2. For $t = N-1, N-2, \ldots, 0$

   For $j \in \{0, 1, \ldots, 2^t - 1\}$

   I. Compute $r_t(j)$ as in (3.11)

   II. Compute

   $$\alpha_t(j) = \frac{\widehat{V}_{t+1}(2j+1) - \widehat{V}_{t+1}(2j)}{su^{r_t(j)}d^{t-r_j(j)}(u-d)}$$

   $$q_t^1(j) = \frac{(1+R)e^{\lambda\alpha_t(j)} - d}{u-d} \quad \text{and} \quad q_t^0(j) = 1 - q_t^1(j)$$

   $$\widehat{V}_t(j) = \frac{1}{(1+R)}\left[q_t^1(j)\widehat{V}_{t+1}(2j+1) + q_t^0(j)\widehat{V}_{t+1}(2j)\right]e^{\beta\alpha_t(j)}$$

   End

End

3. For $t = N, \ldots, 0$

   For $j \in \{0, \ldots, 2^t - 1\}$

   Compute

$$V_t(j) = \widehat{V}_t(j) e^{\beta \sum_{k=0}^{t-1} \alpha_k \left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor \right)}$$

   End

   End

4. For $j \in \{0, 1, \ldots, 2^N - 1\}$ compute $K(j) = K e^{\beta \sum_{k=0}^{N-1} \alpha_k \left(\left\lfloor \frac{j}{2^{N-k}} \right\rfloor \right)}$

   End

3.2.5 Valuation of an Option under Price Impacts

Finally, with the results in the previous Theorem 3.4, we can now explicitly compute the value of an option as follows:

Proposition 3.5. *The price of the derivative security equals*

$$\begin{aligned} V_0(0) &= \tfrac{M}{(1+R)^N} \sum_{j=0}^{2^N-1} \left[e^{\beta \sum\limits_{k=0}^{N-1} \alpha_k(\lfloor j/2^{N-k} \rfloor)} \Phi\left(s u^{r_N(j)} d^{N-r_N(j)} \right) \prod_{k=1}^{N} q_{k-1}^{\bmod (\lfloor j/2^{N-k} \rfloor, 2)} \left(\lfloor j/2^{N-(k-1)} \rfloor \right) \right] \\ &= \tfrac{1}{(1+R)^N} \sum_{j=0}^{2^N-1} \left[V_N(j) \prod_{k=1}^{N} q_{k-1}^{\bmod (\lfloor j/2^{N-k} \rfloor, 2)} \left(\lfloor j/2^{N-(k-1)} \rfloor \right) \right], \end{aligned} \tag{3.29}$$

where $\bmod (c, d)$ *stands for the modulo operation which returns the remainder of the division* c/d.

Proof. We will prove that the formula

$$V_0(0) = \frac{1}{(1+R)^t} \sum_{j=0}^{2^t-1} \left[V_t(j) \prod_{k=1}^{t} q_{k-1}^{\bmod (\lfloor j/2^{t-k} \rfloor, 2)} \left(\lfloor j/2^{t-(k-1)} \rfloor \right) \right] \tag{3.30}$$

holds for every $1 \leq t \leq N$. In order to achieve our goal, we will use induction and prove that (3.30) holds for $t = 1$.

Recall (3.16),

$$V_t(j) \quad = \quad \frac{1}{1+R} \Big[q_t^1(j) V_{t+1}(2j+1) + q_t^0(j) V_{t+1}(2j) \Big].$$

If we substitute $t = 0$ in the previous equation, we obtain

$$V_0(0) = \frac{1}{1+R} \left[q_0^1(0) V_1(1) + q_0^0(0) V_1(0) \right]. \tag{3.31}$$

Remark 3.6. *We use $t = 0$ in order to get $V_0(0)$ so we can get an expression for left hand side of* (3.30)*, but we return to the original value $t = 1$ in the next steps.*

Now we use the value $t = 1$ and for $j = 0$, there are two elements in the sum, say $2j + 1 = 1$ and $2j = 0$. We define $\ell \in \{0, 1\}$ and compute, for $k = 1$, the following expressions

$$\frac{\ell}{2^{t-k}}, \quad \left\lfloor \frac{\ell}{2^{t-k}} \right\rfloor, \quad \frac{\ell}{2^{t-(k-1)}}, \quad \left\lfloor \frac{\ell}{2^{t-(k-1)}} \right\rfloor, \quad \text{and} \quad \bmod \left(\left\lfloor \frac{\ell}{2^{t-k}} \right\rfloor, 2 \right).$$

For $\ell = 0$, those values are respectively

$$\frac{0}{2^{1-1}} = 0, \left\lfloor \frac{0}{2^{1-1}} \right\rfloor = 0, \quad \frac{0}{2^{1-(1-1)}} = 0, \quad \left\lfloor \frac{0}{2^{1-(1-1)}} \right\rfloor = 0, \text{ and } \quad \bmod \left(\left\lfloor \frac{0}{2^{1-1}} \right\rfloor, 2 \right) = 0,$$

whilst for $\ell = 1$, they are

$$\frac{1}{2^{1-1}} = 1, \quad \left\lfloor \frac{1}{2^{1-1}} \right\rfloor = 1, \quad \frac{1}{2^{1-(1-1)}} = \frac{1}{2}, \quad \left\lfloor \frac{1}{2^{1-(1-1)}} \right\rfloor = 0, \text{ and } \quad \bmod \left(\left\lfloor \frac{1}{2^{1-1}} \right\rfloor, 2 \right) = 1.$$

Thus we can express (3.31) as

$$\begin{aligned} V_0(0) &= \frac{1}{1+R} \left[{q_{1-1}}^{\bmod (\lfloor 1/2^{1-1} \rfloor, 2)} \left(\left\lfloor \frac{1}{2^{1-(1-1)}} \right\rfloor \right) V_1(1) + {q_{1-1}}^{\bmod (\lfloor 0/2^{1-1} \rfloor, 2)} \left(\left\lfloor \frac{0}{2^{1-(1-1)}} \right\rfloor \right) V_1(0) \right] \\ &= \frac{1}{1+R} \sum_{j=0}^{2^1-1} \left[V_1(j) {q_{1-1}}^{\bmod (\lfloor j/2^{1-1} \rfloor, 2)} \left(\left\lfloor \frac{j}{2^{1-(1-1)}} \right\rfloor \right) \right], \end{aligned}$$

which shows that (3.30) is satisfied for our base of induction. Henceforth, we assume that (3.30) holds for $1 \le t < N$ in order to prove for $t+1$.

Due to (3.16), we can express

$$V_t(j) = \frac{1}{1+R} \Big(q_t^1(j) V_{t+1}(2j+1) + q_t^0(j) V_{t+1}(2j) \Big),$$

and write (3.30) as

$$V_0(0) = \frac{1}{(1+R)^{t+1}} \sum_{j=0}^{2^t-1} \left[\Big(q_t^1(j) V_{t+1}(2j+1) + q_t^0(j) V_{t+1}(2j) \Big) \prod_{k=1}^{t} {q_{k-1}}^{\bmod (\lfloor j/2^{t-k} \rfloor, 2)} \left(\left\lfloor \frac{j}{2^{t-(k-1)}} \right\rfloor \right) \right]. \tag{3.32}$$

We observe that for each $j \in \{0, \ldots, 2^t - 1\}$, there are two new elements, $2j+1$ and $2j$, added to the sum. This gives a total of $2(2^t) = 2^{t+1}$ elements from 0 to $2^{t+1} - 1$, namely

$$\{0, 1, \ldots, \ 2j, \ 2j+1, \ldots, \ 2(2^t - 1), \ 2(2^t - 1) + 1\} = \{0, 1, \ldots, \ 2j, \ 2j+1, \ldots, \ 2^{t+1} - 2, \ 2^{t+1} - 1\}.$$

Now, we index each of these elements with the letter ℓ, and observe that

$$\begin{array}{llll} \text{for } \ell = 2j+1 & \frac{2j+1}{2^{(t+1)-(t+1)}} = 2j+1 & \Rightarrow & \bmod \left(\left\lfloor \frac{2j+1}{2^{(t+1)-(t+1)}} \right\rfloor, 2 \right) = \bmod (2j+1, 2) = 1 \\ & \frac{2j+1}{2^{(t+1)-(t+1-1)}} = \frac{2j+1}{2} & \Rightarrow & \left\lfloor \frac{2j+1}{2^{(t+1)-(t+1-1)}} \right\rfloor = j \\ \text{for } \ell = 2j & \frac{2j}{2^{(t+1)-(t+1)}} = 2j & \Rightarrow & \bmod \left(\left\lfloor \frac{2j}{2^{(t+1)-(t+1)}} \right\rfloor, 2 \right) = \bmod (2j, 2) = 0 \\ & \frac{2j}{2^{(t+1)-(t+1-1)}} = \frac{2j}{2} = j & \Rightarrow & \left\lfloor \frac{2j}{2^{(t+1)-(t+1-1)}} \right\rfloor = j. \end{array}$$

Then, we can express (3.32) as

$$\begin{aligned} V_0(0) &= \frac{1}{(1+R)^{t+1}} \sum_{j=0}^{2^t-1} \left[\left\{ {q_{t+1-1}}^{\bmod \left(\left\lfloor \frac{2j+1}{2^{(t+1)-(t+1)}} \right\rfloor, 2 \right)} \left(\left\lfloor \frac{2j+1}{2^{(t+1)-(t+1-1)}} \right\rfloor \right) V_{t+1}(2j+1) \right. \right. \\ &\quad \left. \left. + {q_{t+1-1}}^{\bmod \left(\left\lfloor \frac{2j}{2^{(t+1)-(t+1)}} \right\rfloor, 2 \right)} \left(\left\lfloor \frac{2j}{2^{(t+1)-(t+1-1)}} \right\rfloor \right) V_{t+1}(2j) \right\} \prod_{k=1}^{t} {q_{k-1}}^{\bmod (\lfloor j/2^{t-k} \rfloor, 2)} \left(\left\lfloor \frac{j}{2^{t-(k-1)}} \right\rfloor \right) \right] \\ &= \frac{1}{(1+R)^{t+1}} \sum_{\ell=0}^{2^{t+1}-1} \left[V_{t+1}(\ell) \prod_{k=1}^{t+1} {q_{k-1}}^{\bmod (\lfloor \ell/2^{t+1-k} \rfloor, 2)} \left(\left\lfloor \frac{\ell}{2^{t+1-(k-1)}} \right\rfloor \right) \right]. \end{aligned}$$

Hence we have shown (3.30), and by substituting $t = N$ we arrive to the desired formula (3.29). □

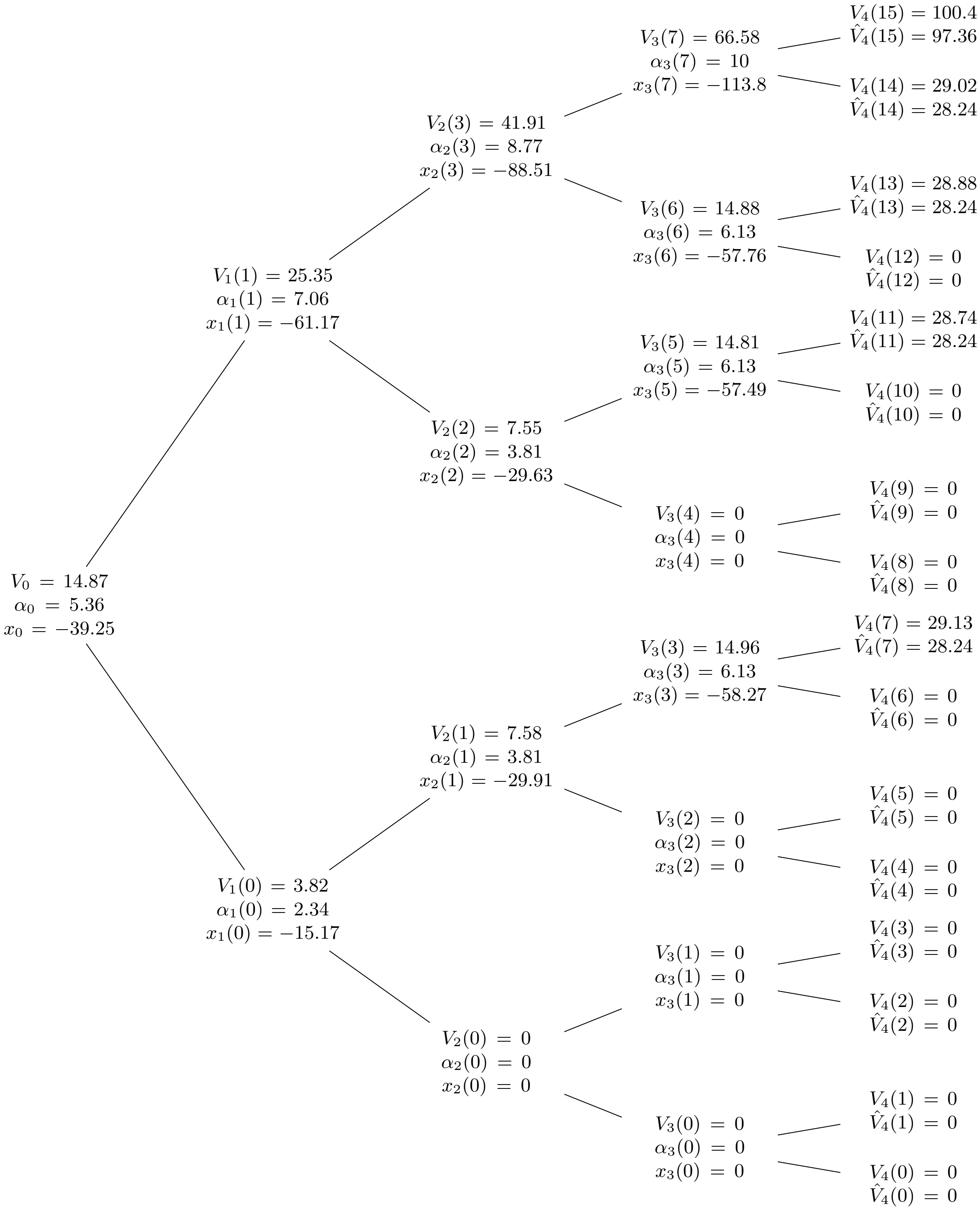
Example 3.7. *Following our previous Example 3.3, we now set* $s = 10$, $u = 1.2$, $d = 0.8$, $\lambda = 0.01$, $\beta = 0.01$ *and* $R = 0$. *If we set* $N = 4$ *and, as before, we assume that the contract is a call with strike price* $K = 11$ *and that the writer of the contract sells* $M = 10$ *contracts, the price of the derivative without assuming price impacts is*

$$\hat{V}_0 = 13.145,$$

whilst if we assume price impacts, the price of the derivative is

$$V_0 = 14.875.$$

We find all the values $V_t(j)$, $\widehat{V}_t(j)$, $\alpha_t(j)$, $x_t(j)$ $q_t^i(j)$, *for* $0 \le t \le N$, $0 \le j \le 2^j - 1$, $i \in \{0, 1\}$. *The next tree shows some values for each node.*



Example 3.8. Tesla Options. *In this example we compute the prices of at-the-money Call and Put of options with underlying TSLA, one month of expiration, reference strike* $K = 180$*, and number of contracts* $M = (500, 2500, 25000)$*. Furthermore we have used the following market impact coefficients* $\lambda = 1 \times 10^{-7}$ *and* $\beta = 1 \times 10^{-8}$.

Strike spread		
	Put (min, max)	*Call* (min, max)
$M = 500$	$(179.9881, 179.9987)$	$(180.0016, 180.0123)$
$M = 5000$	$(179.8805, 179.9871)$	$(180.0164, 180.1236)$
$M = 25000$	$(179.3978, 179.9285)$	$(180.0940, 180.6407)$

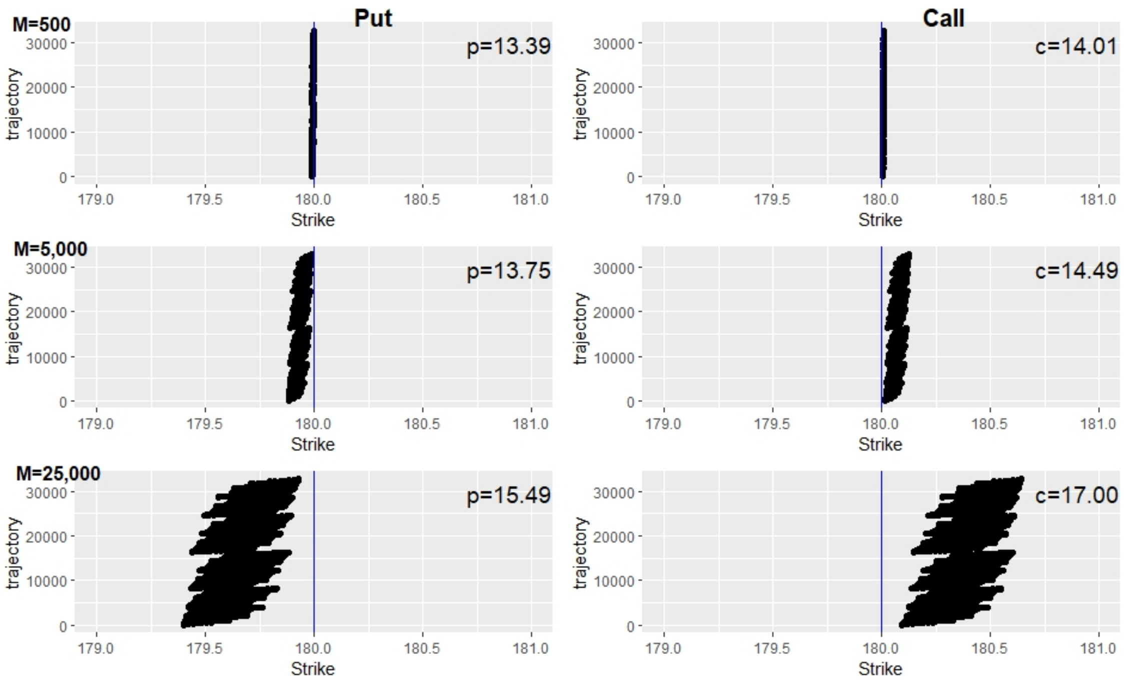


Figure 1: Description of the variability of the strike price K as a function of M. Source: Authors' elaboration

4 Concluding remarks

In this work we analyzed the pricing of contingent claims subject to the influence of price impacts in a discrete-time scenario, and provide explicit formulas in Proposition 3.4 and Corollary 3.5. We observed that trading a large position of derivatives can cause an impact on both the price of a derivative and its underlying asset. Furthermore, we suggest that the valuation of derivatives should take into account the market impacts. In particular, the strike should depend on the size of the trade.

Some open questions to be considered are the following: the model proposed is parametric, how could we estimate those parameters? can we make a similar analysis in continuous-time models by using stochastic calculus? Can this model give more clues regarding the volatility smile?

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A One period case

We set a market that consists of two assets, a bond and a stock. The price of the bond is denoted by B_t, with values,

$$\begin{aligned} B_0 &= 1, \\ B_1 &= 1+R, \end{aligned}$$

where $R > 0$ refers to the one-period interest rate. The stock is known as a random asset because its price S_t at $t = 1$ can not be anticipated at $t = 0$. For instance, at time $t = 0$ the price is observed as $S_0 = s$, but at the future time $t = 1$ we have that

$$S_1 = sZ \qquad \text{where} \quad Z = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p \end{cases},$$

where $d \leq 1 + R \leq u$.

In this market, a *portfolio* $h = (x, \alpha)$ is a vector in $\mathbb{R}^2$, where the component x stands for the number of bonds, and α represents the number of stocks acquired by the trader (recall that $\alpha > 0$ means a purchase of stocks, consequently, $\alpha < 0$ means a sale of stocks). The *value process* (or simply the *value*) of the portfolio $h = (x, \alpha)$ is

$$\begin{aligned} \bar{V}_0^h &= x + \alpha s, \\ \bar{V}_1^h &= x(1+R) + \alpha s Z. \end{aligned}$$

Now we introduce a *contingent claim* or *financial derivative* for this market as a random variable X of the form $X = \Phi(Z)$, where Z is the random variable described above. This claim is interpreted as an agreement between two agents: the holder (also known as the owner) and the underwriter. Both establish that the holder receives (or must pay) a quantity X at a prescribed time t (in this case $t = 1$). The function Φ is called the *contract function.*

An example of a financial derivative is the well-known European option. Namely, M-units of a European option on a unit of stock, with strike price K and exercise date N (in this case $N = 1$) establishes that at maturity time $t = N$, the holder has the right, but not the obligation, to buy or sell M units of stock (one for each contract), called the *underlying*, at the price K, to the underwriter. The price K is set at time $t = 0$. We will consider two types of European options, known as call and put options (or simply call and put), respectively. The call option gives the right to buy the underlying security, while the put option gives the right to sell it. The situation of interest (but not the only one) is, of course,

$$sd < K < su. \tag{A.1}$$

If the contract is a call and at exercise date $N = 1$, K is less than the actual price S_1, the option will be executed, otherwise, the holder will let it expire. Observe that if it is exercised at time $t = 1$, the holder pays K to the underwriter in exchange for the stock. Thus, the payoff or contract function of this claim is

$$X = \Phi(sZ) := \max\{sZ - K, 0\}. \tag{A.2}$$

In the case of put options, we have an analogous situation. Now the holder will execute his option only if the price at time $t = 1$, S_1, is below the strike price K and therefore the contract function of the claim is

$$X = \Phi(sZ) := \max\{K - sZ, 0\}. \tag{A.3}$$

It follows that if there exists a portfolio h such that $\bar{V}_1^h = X$, with probability 1, then h is called a *replicating portfolio* of X. This means that regardless of the outcome Z at $t = 1$, the value of h will be the same as X, with probability 1.

Some standard results in the one-period binomial model are that for a replication portfolio $h = (x, \alpha)$, we have

$$\alpha = \frac{M}{s}\frac{\Phi(su) - \Phi(sd)}{u - d} \tag{A.4}$$

$$x = \frac{M}{1+R}\frac{u\Phi(sd) - d\Phi(su)}{u - d} \tag{A.5}$$

and therefore, the price of the contingent claim X is given by the value at $t = 0$ of the replicating portfolio h. In other words, the price of the premium is (see for instance Chapter 2, in Björk [5])

$$\bar{V}_0^h = \frac{M}{1+R}\left[\left\{\frac{(1+R) - d}{u - d}\right\}\Phi(su) + \left\{\frac{u - (1+R)}{u - d}\right\}\Phi(sd)\right]. \tag{A.6}$$

Remark A.1. *1. In fact, the underwriter hedges his position by holding the replicating portfolio $h = (x, \alpha)$. Thereby, at time $t = 0$ the underwriter buys x bonds, and α units of stock according to* (A.1)–(A.5).

2. The previous formulae work even in the naive cases when (A.1) *are not satisfied, namely if $K < sd < su$ or $sd < su < K$. For example, for a call option with $K < sd < su$, we have*

$$\alpha = M, \qquad x = -\frac{MK}{1+R}, \quad \text{and} \tag{A.7}$$

$$\bar{V}_0^h = Ms - \frac{MK}{1+R}. \tag{A.8}$$

B Deriving the equations for the Algorithm

For this section, we assume that $t \in \{0, \dots, N\}$ and $j \in \{0, \dots, 2^t - 1\}$, and use the following fact: for $n \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $h \in \mathbb{N}$, we have

$$\left\lfloor \frac{2n+1}{2^h} \right\rfloor = \left\lfloor \frac{2n}{2^h} + \frac{1}{2^h} \right\rfloor = \left\lfloor \frac{2n}{2^h} \right\rfloor = \left\lfloor \frac{n}{2^{h-1}} \right\rfloor.$$

If $k \in \{0, \dots, t\}$, then $t + 1 - k > 0$ and

$$\sum_{k=0}^{t} \alpha_k \left(\left\lfloor \frac{2j+1}{2^{t+1-k}} \right\rfloor \right) = \sum_{k=0}^{t} \alpha_k \left(\left\lfloor \frac{2j}{2^{t+1-k}} \right\rfloor \right) = \sum_{k=0}^{t} \alpha_k \left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor \right)$$

Let us recall equation (3.23):

$$\widehat{V}_t(j) := V_t(j) e^{-\beta \sum_{k=0}^{t-1} \alpha_k \left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor \right)} \tag{B.1}$$

In order to obtain (3.24), first of all, we observe that for $t = N$

$$
\begin{aligned}
\widehat{V}_N(j) &= V_N(j) e^{-\beta \sum\limits_{k=0}^{N-1} \alpha_k\left(\left\lfloor \frac{j}{2^{N-k}} \right\rfloor\right)} \\
&= M\Phi\left(s u^{r_N(j)} d^{N-r_N(j)} e^{\beta \sum\limits_{k=0}^{N-1} \alpha_k\left(\left\lfloor \frac{j}{2^{N-k}} \right\rfloor\right)} \right) e^{-\beta \sum\limits_{k=0}^{N-1} \alpha_k\left(\left\lfloor \frac{j}{2^{N-k}} \right\rfloor\right)} \\
&= M e^{\beta \sum\limits_{k=0}^{N-1} \alpha_k\left(\left\lfloor \frac{j}{2^{N-k}} \right\rfloor\right)} \Phi\left(s u^{r_N(j)} d^{N-r_N(j)} \right) e^{-\beta \sum\limits_{k=0}^{N-1} \alpha_k\left(\left\lfloor \frac{j}{2^{N-k}} \right\rfloor\right)} \\
&= M\Phi\Big(s u^{r_N(j)} d^{N-r_N(j)} \Big),
\end{aligned} \tag{B.2}
$$

where we have used equation (3.15). Second of all, by solving $V_t(j)$ from (B.1) we get

$$
V_t(j) = \widehat{V}_t(j) e^{\beta \sum\limits_{k=0}^{t-1} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)}, \tag{B.3}
$$

which is equation (3.28). Third of all, from (3.14), (3.18), and (B.3), we have

$$
\begin{aligned}
\alpha_t(j) &= \frac{V_{t+1}(2j+1) - V_{t+1}(2j)}{\widehat{S}_t(j)(u-d)} \\
&= \frac{V_{t+1}(2j+1) - V_{t+1}(2j)}{s u^{r_t(j)} d^{t-r_t(j)} e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)} (u-d)} \\
&= \frac{\widehat{V}_{t+1}(2j+1) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{2j+1}{2^{t+1-k}} \right\rfloor\right)} - \widehat{V}_{t+1}(2j) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{2j}{2^{t+1-k}} \right\rfloor\right)}}{s u^{r_t(j)} d^{t-r_t(j)} (u-d) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)}} \\
&= \frac{\widehat{V}_{t+1}(2j+1) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)} - \widehat{V}_{t+1}(2j) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)}}{s u^{r_t(j)} d^{t-r_t(j)} (u-d) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)}} \\
&= \frac{\widehat{V}_{t+1}(2j+1) - \widehat{V}_{t+1}(2j)}{s u^{r_t(j)} d^{t-r_t(j)} (u-d)},
\end{aligned}
$$

which is equation (3.26). If we let $t = N - 1$, we obtain equation (3.24) .Finally, in order to obtain (3.27) and (3.25), we use again (B.3), and rewrite (3.16)

$$
\begin{aligned}
V_t(j) &= \frac{1}{1+R}\Big[q_t^1(j) V_{t+1}(2j+1) + q_t^0(j) V_{t+1}(2j) \Big] \\
&= \frac{1}{1+R}\Big[q_t^1(j) \widehat{V}_{t+1}(2j+1) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{2j+1}{2^{t+1-k}} \right\rfloor\right)} + q_t^0(j) \widehat{V}_{t+1}(2j) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{2j}{2^{t+1-k}} \right\rfloor\right)} \Big] \\
&= \frac{1}{1+R}\Big[q_t^1(j) \widehat{V}_{t+1}(2j+1) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)} + q_t^0(j) \widehat{V}_{t+1}(2j) e^{\beta \sum_{k=0}^{t} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)} \Big] \\
&= \frac{1}{1+R}\Big[q_t^1(j) \widehat{V}_{t+1}(2j+1) + q_t^0(j) \widehat{V}_{t+1}(2j) \Big] e^{\beta \sum_{k=0}^{t-1} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right) + \beta \alpha_t(j)}.
\end{aligned}
$$

This implies that

$$
\widehat{V}_t(j) = V_t(j) e^{-\beta \sum_{k=0}^{t-1} \alpha_k\left(\left\lfloor \frac{j}{2^{t-k}} \right\rfloor\right)} = \frac{1}{1+R}\Big[q_t^1(j) \widehat{V}_{t+1}(2j+1) + q_t^0(j) \widehat{V}_{t+1}(2j) \Big] e^{\beta \alpha_t(j)}.
$$

which is equation (3.27), and if we substitute $t = N - 1$, we get equation (3.25).

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