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Abstract

In this paper we provide conditions under which the hitting-time problem for Brownian motion is equivalent to solving a heat equation with moving boundary and distributional initial conditions. Motivated by the hitting-time problem, we study the heat equation with absorbing moving boundaries. Using Fourier analysis we develop a procedure to solve this problem for a family of curves that includes the square root, quadratic, and cubic boundaries. As an application of our results, and using Sturm-Liouvile theory, we compute the density of the hitting time of a Brownian motion to a family of quadratic boundaries.

Keywords: Heat equation, Brownian motion, Hitting times, Sturm-Liouville theory

2010 MSC: 35K05, 35K08, 60J65, 60G40

1. Introduction

Consider a standard one-dimensional Brownian motion $\{B_t : 0 \leq t < \infty\}$ starting at zero, and a real-valued continuous function f on $[0, \infty)$. We define the first hitting time as

$$T_f := \inf\{t > 0 : B_t = f(t)\}. \tag{1}$$

The first hitting time problem is then to find the distribution of T_f. Hitting time problems are also known as boundary crossing problems and they are fundamental and challenging problems in stochastic analysis. The study of hitting time problems may be traced back to Bachelier's doctoral thesis [1] and nowadays they have deep applications in pure and applied mathematics.

In this paper we introduce a new approach for solving the boundary crossing problem for Brownian motion for certain smooth and convex boundaries. As a first contribution of the present paper, in Section 2, using some results from [2], we give conditions under which the boundary crossing problem for Brownian motion is equivalent to solving a heat equation with moving boundary and distributional initial conditions. This result can be seen as a complement to the celebrated method of images developed by Lerche [3] that works for concave and sub-linear boundaries.

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The study of hitting time problems for Brownian motion leads us to deal with the heat equation absorbed at a moving boundary $f(t)$. Thus, we wish to find functions $v(t, x)$ such that

$$\nu_t(t,x) = \frac{1}{2}\nu_{xx}(t,x), \tag{2}$$

$$\nu(t, f(t)) = 0, \tag{3}$$

$$(t,x) \in [0,\infty) \times \mathbb{R}, \quad f \in C^2.$$

Although (2)-(3) seems to be a particular problem in stochastic analysis, it actually appears prominently in applications. For instance, see [4, 2, 5, 6] in the construction of first hitting time densities of Brownian motion; [7, 8, 9] in the valuation of barrier options; [10, 11] in the quantification of counterparty risk; [12, 13] for applications of the quadratic boundary in biology and other fields; [14, 15] for applications in physics. In fact, explicit solutions to the problem (2)-(3) are well known in some particular cases, such as the followings.

(a) *Linear boundary.* For $b \in \mathbb{R}$, the function

$$\nu(t,x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{x^2}{2t}\right\} + b\frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}. \tag{4}$$

solves (2)-(3) in the case in which $f(t) = -bt$. [See, for instance, [16] for an example of this function in the first hitting time of Brownian motion to a linear boundary.]

(b) *Quadratic boundary.* Given that A_i denotes an Airy function and $\xi \in \mathbb{R}^- := \{x \in \mathbb{R}; x < 0\}$ is any of its roots, then

$$\nu(t,x) = \exp\left\{\frac{t^3}{12} + \frac{tx}{2}\right\} A_i\left(x + \frac{t^2}{4}\right) \tag{5}$$

is a solution of problem (2)-(3) when $f(t) = \xi - t^2/4$. (See, for instance, [17] for the general theory and applications of Airy functions)

(c) *Rayleigh type equation.* Let

$$\nu(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{i\lambda x - \frac{1}{2}\lambda^2 t - \frac{\lambda^4}{4}\right\} d\lambda$$

be the so-called Pearcey function. This function solves problem (2)-(3) when f satisfies

$$f''(t) = 2[f'(t)]^3 - \frac{1}{2}tf'(t) - \frac{1}{4}f(t).$$

See [18].

As a second contribution of the present paper, using basic elements of Fourier analysis, we develop a unified procedure to solve problem (2)-(3) for a family of boundaries that include quadratic and cubic boundaries. The technique used to achieve our goal is remarkably straightforward, and is based on analyzing the convolution between the fundamental solution of the heat equation and some real-valued and sufficiently smooth function ϕ. We suspect that this technique can be generalized to the case of a general polynomial boundary.

Finally, given that our main motivation for studying problem (2)-(3) comes from the hitting time problem for Brownian motion, we use our results to compute the density of the boundary crossing up to a quadratic boundary. To this end, we use the solution to the heat equation with quadratic moving boundary computed in Section 4, below, and Sturm-Liouville theory.

Organization of the paper. In Section 2 we show that, for certain smooth and convex boundaries, the boundary crossing problem for Brownian motion is equivalent to solving the heat equation with distributional initial and moving boundary conditions. In Section 3 we introduce some notation, define the heat polynomials, and recall some of their properties. In Section 4 the technique used to link solutions ν of the heat equation with moving boundaries f is introduced in the case in which the linking function $\phi(x)$ has Fourier transform $\overline{\phi}$ which is an entire function with growth $(2, \sigma)$. Furthermore, in Section 5 we study in detail the case of absorption at the linear, quadratic, and square root boundaries with our approach. Subsequently, in Section 6, we derive the solution of the heat equation with a cubic absorbing boundary. As an application of our results, in Section 7 we compute the density of the hitting time up to a family of quadratic boundaries. Finally, we conclude in Section 8 with some general remarks.

2. On the heat equation and hitting-time problems

In this section we present a brief and self-contained summary of [2] and give conditions under which the boundary crossing problem for Brownian motion is equivalent to solving a heat equation with suitable initial and moving boundary conditions. The main goal in this section is to correct an erratum that appeared in the boundary conditions (6.5) in [2].

The use of the heat equation with a moving boundary for finding the density of a hitting time appeared for the first time in the method of images developed by Lerche [3] Theorem 1.1. A concise statement of this theorem appears in Proposition 3.3 in [19]. Among the type of boundaries ψ for which Lerche's approach holds are:

1. ψ is a concave function,
2. $\psi(t)/t$ is monotone decreasing that is, ψ is a sub-linear function.

Consider a boundary $\psi \in C^{\infty}[0, \infty)$ that is concave and sub-linear on a region $\Omega := \{(t, x) : x \leq \psi(t)\}$. Roughly speaking, the method of images states that if you have a solution h to the heat equation on Ω that satisfies

1. $h(t, \psi(t)) = 0$;
2. $\lim_{t\downarrow 0} h(t, a) = \delta_0(a)$,

then the density of the hitting time up to the boundary ψ is

$$f^{\psi}(t) = -\frac{1}{2} h_x(t, \psi(t)).$$

The theory developed in this section can be seen as a complement to the Lerche's method of images. In fact, we characterize the density of hitting times of Brownian motion as solutions to the heat equation with distributional initial and moving boundary conditions, considering boundaries that do not fit Lerche's approach. We study the hitting times for convex boundaries (see Assumption 2.1 below) that include, for instance, polynomial boundaries such as $f(t) = a + bt^n$ for $a, b > 0$ and n a positive integer.

Recall that given a standard one-dimensional Brownian motion $\{B_t : 0 \leq t < \infty\}$ starting at zero, and a real-valued continuous function f on $[0, \infty)$ and such that $f(0) \neq 0$, the hitting time problem is to find the distribution of the random time

$$T_f := \inf\{t > 0 : B_t = f(t)\}. \tag{6}$$

In the case of a Brownian motion it is known that if f is a C^1 function, then T_f admits a continuous density with respect to the Lebesgue measure [20].

Consider first the hitting time to a fixed level a, so

$$T_a := \inf\{t > 0 : B_t = a\}.$$

Using the reflection principle (see [16] p. 81) it can be shown that, for $a > 0$ fixed, T_a has a density with respect to the Lebesgue measure given by

$$h(t, a) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}} \quad \text{for} \quad t > 0. \tag{7}$$

We will study the limits of the function h because they will play a key role in determining the boundary conditions (22) and (21) below. Note that if $\omega(t, a)$ is the heat kernel (also known as the fundamental solution to the heat equation), that is,

$$\omega(t, a) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}}, \tag{8}$$

then $\omega_a(t, a) = -h(t, a)$ where h is as in (7), and ω_a denotes partial derivative.

Let $C_c^\infty(\mathbb{R})$ be the set of infinitely differentiable functions with compact support, and recall that ([21] p. 208)

$$\lim_{t \downarrow 0} \omega(t, a) = \delta_0(a),$$

where $\delta_0(a)$ is the Dirac mass at 0 with respect to the variable a. Then, given that differentiation is continuous with respect to distributional convergence ([21] p. 315), we have

$$\lim_{t \downarrow 0} h(t, a) = -\frac{1}{2} \delta_0'(a) \tag{9}$$

where the limit is in distributional sense, that is,

$$\lim_{t \downarrow 0} \int_0^\infty h(t, a) \varphi(a) da = \frac{1}{2} \varphi'(0) \tag{10}$$

for all φ in the set $\mathcal{E}$ of test functions defined as

$$\mathcal{E} := \{\varphi \in C_c^\infty([0, \infty)) : \varphi(0) = 0\}. \tag{11}$$

Besides, direct calculations show that

$$\lim_{a \downarrow 0} h(t, a) = 0. \tag{12}$$

We will study the hitting time problem (6) for a function f that satisfies the following conditions.

Assumption 2.1. *Let $f \in C^2([0, \infty))$ be a real-valued function such that, for all $t > 0$,*

$$f(0) = 0, \quad f''(t) \geq 0, \quad \textit{and} \quad \int_0^t (f'(s))^2 ds < \infty. \tag{13}$$

Remark 2.2. *We will denote by* $\nu_f(t,a)$ *the density of the hitting time for a boundary* $a+f(t)$ *for* $a>0$ *and such that* f *satisfies Assumption 2.1.*

Theorem 3.1 together with Theorem 4.1 in [2] show that the density of the hitting time $\nu_f(s,a)$ is given by

$$\nu_f(s,a) = v(0,a)e^{-\frac{1}{2}\int_0^s (f'(u))^2du - f'(0)a}h(s,a) \tag{14}$$

with h as in (7), and $v:[0,s]\times[0,\infty)\to[0,\infty)$ is a solution to the Cauchy problem

$$-\frac{\partial v}{\partial t} + f''(t)av = \frac{1}{2}\frac{\partial^2 v}{\partial a^2} + \Big(\frac{1}{a} - \frac{a}{s-t}\Big)\frac{\partial v}{\partial a} \quad \text{with} \quad v(s,a)=1, \tag{15}$$

as well as $0\le v(t,a)\le 1$ for $0\le t\le s$.

To obtain a solution v to (15), Proposition 4.1 in [2] considers a function w defined as

$$v(t,a) = \frac{w(t,a)}{h(s-t,a)}, \tag{16}$$

where h is given in (7). If $w(t,x)$ satisfies that

$$-w_t(t,a) + f''(t)aw(t,a) = \frac{1}{2}\frac{\partial^2 w}{\partial a^2} \quad \text{on} \quad [0,s)\times(0,\infty), \tag{17}$$

then $v(t,a)$ in (16) satisfies (15). Furthermore, from the boundary condition (9) and the fact that $v(s,a)=1$ we obtain

$$\lim_{t\uparrow s} w(t,a) = -\frac{1}{2}\delta_0'(a) \tag{18}$$

and, from (12) it follows that

$$\lim_{a\downarrow 0} w(t,a) = 0. \tag{19}$$

Using Fourier transforms, Theorem 6.1 in [2] establishes that a solution to (17) on $[0,s)\times(0,\infty)$ is

$$w(t,a) = e^{\frac{1}{2}\int_t^s (f'(u)^2du + af'(t)}\kappa(s-t, a+\int_t^s f'(u)du) \tag{20}$$

where $\kappa(t,x)$ is a solution to the heat equation.

Finally, for a solution κ to the heat equation and considering the initial condition (18) we obtain

$$\lim_{t\uparrow s}\kappa\Big(s-t, a+\int_t^s f'(u)du\Big) = \lim_{t\downarrow 0}\kappa(t,a) = -\frac{1}{2}\delta_0'(a) \tag{21}$$

where the limit is in the distributional sense on the set of test functions $\mathcal{E}$ in (11).

The condition (19) yields

$$\lim_{a\downarrow 0}\kappa\Big(s-t, a+\int_t^s f'(u)du\Big) = \kappa(s-t, f(s)-f(t)) = 0. \tag{22}$$

The Condition 6.5 in [2] has to be replaced by the conditions (21) and (22).

Summarizing, if κ is a solution to the heat equation on $[0,s)\times(0,\infty)$ that satisfies (21) and (22), then the density of the hitting time at time s, $\nu_f(s,a)$, is

$$\nu_f(s,a) = \kappa\Big(s, a+\int_0^s f'(u)du\Big) = \kappa(s, a+f(s)). \tag{23}$$

Example 2.3 (The hitting time up to a linear boundary)**.** *This example is devoted to show that the hitting time up to the linear boundary* $f(t,a) = a + \mu t$ $(\mu > 0,\ a > 0)$ *can be found with a solution to the heat equation that satisfies (21) and (22) with* $f(t) = \mu t$*. To this end, consider the function* $\kappa(t,x)$ *given by*

$$\kappa(t,x) = \frac{(x-\mu t)}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} = h(t,x) - \omega(t,x), \tag{24}$$

where h *was given in (7) and* ω *is the heat kernel (8). Given that* h *and* ω *both are solutions to the heat equation, it follows that also* κ *solves the heat equation. Furthermore, if* $\varphi \in \mathcal{E}$*, then*

$$\lim_{t \downarrow 0} \int_0^{\infty} \varphi(a)\kappa(t,a)da = \frac{1}{2}\varphi(0) + \frac{1}{2}\varphi'(0) = \frac{1}{2}\varphi'(0) = -\frac{1}{2}\delta_0'(a),$$

and so κ *satisfies (21). Finally, if* s *is a fixed number, it follows that*

$$\kappa(s-t, f(s)-f(t)) = \frac{(\mu s - \mu t - \mu(s-t))}{\sqrt{2\pi(s-t)^3}} e^{-\frac{(\mu s-\mu t)^2}{2(s-t)}} = 0.$$

Hence κ *satisfies (22). Therefore, the density* ν_f *is given by*

$$\nu_f(s,a) = \kappa(s, a+\mu s) = \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a+\mu s)^2}{2s}}. \tag{25}$$

This density can be checked in [16] p. 196 where it is obtained by means of the Girsanov theorem.

Motivated by the hitting time problem for Brownian motion, in the next sections we address the problem of the heat equation with a moving boundary, that is, problem (2)-(3).

3. Technical preliminaries

In this section we introduce notation that will be used in the remainder of the paper. We also define the so-called heat polynomials and state some of their properties.

Now we will define the so-called heat polynomials and present some of their properties. We will make use of heat polynomials in Lemma 3.2 and in the proof of Proposition 4.1 below.

Definition 3.1 (Rosembloom and Widder [22])**.** *We define the heat polynomials as functions* $v_n(t,x)$ *such that*

$$e^{i\lambda x - \lambda^2 t/2} = \sum_{n=0}^{\infty} v_n(t/2, x)\frac{(i\lambda)^n}{n!}.$$

The heat polynomials satisfy the following properties whose proofs can be seen in [22].

- The recurrence relation

$$v_{n+1}(t,x) = x v_n(t,x) + 2nt v_{n-1}(t,x) \tag{26}$$

  holds for $n = 1, 2, \ldots,$ with $v_0(t,x) = 1,\ v_1(t,x) = x$.

- For $n = 1, 2, \ldots,$

$$v_n^{(1)}(t,x) = n v_{n-1}(t,x). \tag{27}$$

Finally, we introduce a technical lemma that will be used in the proof of Proposition 4.1.

Lemma 3.2. *Let $v_n(t,x)$ be the heat polynomials. Then*

$$\frac{d^n}{d\lambda^n}\left[e^{i\lambda x-\lambda^2 t/2}\right] = v_n(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2} \quad \text{for} \quad n = 1, 2\cdots. \tag{28}$$

Proof. The proof is by induction. For $n = 1$, from (26) direct calculations give the result. Now, assume that (28) holds for n; we will prove that it holds for $n+1$. The induction hypothesis yields

$$\begin{aligned}\frac{d^{n+1}}{d\lambda^{n+1}}\left[e^{i\lambda x-\lambda^2 t/2}\right] &= \frac{d}{d\lambda}\left[\frac{d^n}{d\lambda^n}\left[e^{i\lambda x-\lambda^2 t/2}\right]\right] \\ &= \frac{d}{d\lambda}\left[v_n(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}\right],\end{aligned}$$

which together with (27) gives

$$\begin{aligned}\frac{d^{n+1}}{d\lambda^{n+1}}\left[e^{i\lambda x-\lambda^2 t/2}\right] &= \Big[-tnv_{n-1}(-t/2, x-\lambda t) \\ &\quad +v_n(-t/2, x-\lambda t)(ix-\lambda t)\Big]e^{i\lambda x-\lambda^2 t/2} \\ &= v_{n+1}(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}.\end{aligned}$$

The last equality follows from (26). The proof is complete. □

4. Main results

In this section we derive some algebraic properties of the convolution between the fundamental solution of the heat equation and a function ϕ which has Fourier transform $\overline{\phi}$ which in turn, is an entire function of growth $(2,\sigma)$, for more details on this assumption see for instance Theroem 12.1 in [22].

Proposition 4.1. *For positive integers p, q, r and constant coefficients $a, b \in \mathbb{R}$, consider the differential equation*

$$x^p\phi^{(2)}(x) = ax^q\phi^{(1)}(x) + bx^r\phi^{(0)}(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{29}$$

In addition, let v_n be the heat polynomials. If $\overline{\phi}$ is an entire function of growth $(2,\sigma)$ and denotes the Fourier transform of a solution ϕ to (29), then the following holds

$$\begin{aligned}(-i)^p\int(i\lambda)^2\overline{\phi}(\lambda)v_p(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}d\lambda& \\ = (-i)^q a\int i\lambda\overline{\phi}(\lambda)v_q(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}d\lambda& \\ +(-i)^r b\int\overline{\phi}(\lambda)v_r(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}d\lambda.&\end{aligned} \tag{30}$$

Proof. Applying the Fourier transform to both sides of (29) yields

$$i^p\frac{d^p}{d\lambda^p}\left[(i\lambda)^2\overline{\phi}(\lambda)\right] = ai^q\frac{d^q}{d\lambda^q}\left[i\lambda\overline{\phi}(\lambda)\right] + bi^r\frac{d^r}{d\lambda^r}\left[\overline{\phi}(\lambda)\right].$$

Next, we multiply both sides of the previous expression by $e^{i\lambda x-\lambda^2 t/2}$, and then integrate to obtain

$$\int_{-\infty}^{+\infty} e^{i\lambda x-\lambda^2 t/2} i^p \frac{d^p}{d\lambda^p}\left[(i\lambda)^2\overline{\phi}\right] d\lambda \tag{31}$$
$$= a\int_{-\infty}^{+\infty} e^{i\lambda x-\lambda^2 t/2} i^q \frac{d^q}{d\lambda^q}\left[i\lambda\overline{\phi}\right] d\lambda + b\int_{-\infty}^{+\infty} e^{i\lambda x-\lambda^2 t/2} i^r \frac{d^r}{d\lambda^r}\left[\overline{\phi}\right] d\lambda.$$

Note that the function $e^{i\lambda x-\lambda^2 t/2}$ vanishes as $|\lambda| \to \infty$. Hence, integration by parts gives

$$(-i)^p \int (i\lambda)^2 \overline{\phi} \frac{d^p}{d\lambda^p}\left[e^{i\lambda x-\lambda^2 t/2}\right] d\lambda$$
$$= (-i)^q a \int (i\lambda)^1 \overline{\phi} \frac{d^q}{d\lambda^q}\left[e^{i\lambda x-\lambda^2 t/2}\right] d\lambda$$
$$+(-i)^r b \int (i\lambda)^0 \overline{\phi} \frac{d^r}{d\lambda^r}\left[e^{i\lambda x-\lambda^2 t/2}\right] d\lambda.$$

Thus, equation (30) follows from the latter equality and (28). □

We will use Proposition 4.1 in combination with the following facts.

Remark 4.2. *(a) Suppose that there exists a pair of functions ν and f that solve the moving boundary problem (2)-(3). Then direct calculations give*

$$f'(t)\nu^{(1)}(t, f(t)) + \frac{1}{2}\nu^{(2)}(t, f(t)) = 0 \tag{32}$$

and

$$f''(t)\nu^{(1)} + f'(t)\left(f'(t)\nu^{(2)} + \nu^{(3)}\right) + \frac{1}{4}\nu^{(4)} = 0. \tag{33}$$

(b) Consider the function $\nu(t, x)$ defined as the convolution between a solution ϕ to (29) and the fundamental solution to the heat equation, i.e.,

$$\nu(t, x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\phi}(\lambda) e^{i\lambda x-\lambda^2 t/2} d\lambda \quad \textit{for} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

If ϕ satisfies some growth condition (see [16] p. 254), then $\nu(t, x)$ is a solution to the heat equation. Furthermore, by properties of Fourier transforms,

$$\nu^{(n)}(t, x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\lambda)^n \overline{\phi}(\lambda) e^{i\lambda x-\lambda^2 t/2} d\lambda \tag{34}$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

We are now ready to present the main result of this section which uses Proposition 4.1 and Remark 4.2. Theorem 4.3 links the functions ν and f that solve (2)-(3), through a specific C^2 function ϕ.

Theorem 4.3. *For given fixed coefficients $d_0, d_1, c_0, c_1, c_2 \in \mathbb{R}$, let ϕ be a real-valued solution of the following second order ODE*

$$\phi^{(2)}(x) = \sum_{j=0}^{1} d_j x^j \phi^{(1)}(x) + \sum_{j=0}^{2} c_j x^j \phi(x) \tag{35}$$

for $x \in \mathbb{R}$, with Fourier transform $\overline{\phi}$. Let

$$\nu(t,x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\phi}(\lambda) e^{i\lambda x - \lambda^2 t/2} d\lambda \tag{36}$$

be the convolution between ϕ and the fundamental solution of the heat equation. If there exists f such that $\nu(t, f(t)) = 0$ and $\nu^{(1)}(t, f(t)) \neq 0$, it follows that

1. *If $d_1 = c_2 = 0$ in (35), then we have that*

$$f(t) = -\frac{d_0}{2}t - \frac{c_1}{4}t^2. \tag{37}$$

2. *If at least one of the coefficients d_1, c_2 is different from zero, then, for an arbitrary constant $\mathcal{C}$, the function f is of the form*

$$f(t) = \frac{-d_0 d_1 - 2c_1 - 2c_2 d_0 t + c_1 d_1 t}{d_1^2 + 4c_2} + \sqrt{-1 + d_1 t + c_2 t^2} \cdot \mathcal{C}. \tag{38}$$

Proof. If the function ϕ is a solution of (35), it follows from Proposition 4.1 and (34) that its convolution with the fundamental solution of the heat equation satisfies that

$$\begin{aligned}(1 - d_1 t - c_2 t^2)&\nu^{(2)}(t,x) = \\ &(d_0 + d_1 x + c_1 t + c_2 2tx)\nu^{(1)}(t,x) \\ &\quad +(c_0 + c_1 x + c_2 x^2 + c_2 t)\nu^{(0)}(t,x).\end{aligned} \tag{39}$$

Now, suppose that there exists a function f such that $\nu(t, f(t)) = 0$ for all $t \geq 0$. It follows from (39) that

$$\begin{aligned}(1 - d_1 t - c_2 t^2)&\nu^{(2)}(t, f(t)) \\ &= (d_0 + c_1 t + [d_1 + 2c_2 t] f(t))\nu^{(1)}(t, f(t))\end{aligned} \tag{40}$$

and, from (32),

$$\nu^{(2)}(t, f(t)) = -2f'(t)\nu^{(1)}(t, f(t)). \tag{41}$$

Thus, from (40) and (41), we have that

$$-2f'(t)(1 - d_1 t - c_2 t^2) = (d_0 + c_1 t + [d_1 + 2c_2 t] f(t)). \tag{42}$$

This implies that f, which solves the latter ODE, has the following general solution (by standard techniques) as long as at least one of the coefficients d_1, c_2 is not zero

$$f(t) = \frac{-d_0 d_1 - 2c_1 - 2c_2 d_0 t + c_1 d_1 t}{d_1^2 + 4c_2} + \sqrt{-1 + d_1 t + c_2 t^2} \cdot \mathcal{C}.$$

In turn, if $d_1 = c_2 = 0$, it follows from (42) that

$$f(t) = -\frac{d_0}{2}t - \frac{c_1}{4}t^2,$$

as claimed. □

5. Applications of Theorem 4.3

In this section we will show how Theorem 4.3 works to solve (2)-(3).

5.1. The linear boundary

From the proof of Theorem 4.3 we know that if all the coefficients are zero except d_0, then we will recover the linear boundary.

Next we will proceed to construct a solution. To this end recall that, for arbitrary constants C_1 and C_2, the equation

$$\phi^{(2)}(x) = d_0\phi^{(1)}(x) + c_0\phi^{(0)}(x),$$

has the general solution

$$\phi(x) = e^{1/2\left(d_0 - \sqrt{4c_0+d_0^2}\right)x}C_1 + e^{1/2\left(d_0 + \sqrt{4c_0+d_0^2}\right)x}C_2.$$

If we take convolution between this function and the fundamental solution of the heat equation, we obtain

$$\begin{aligned} \nu(t,x) &= e^{1/2\left(d_0 - \sqrt{4c_0+d_0^2}\right)x + \frac{1}{2}t\left[1/2\left(d_0 - \sqrt{4c_0+d_0^2}\right)\right]^2}C_1 \\ &\quad + e^{1/2\left(d_0 + \sqrt{4c_0+d_0^2}\right)x + \frac{1}{2}t\left[1/2\left(d_0 + \sqrt{4c_0+d_0^2}\right)\right]^2}C_2. \end{aligned} \tag{43}$$

Thus if $C_1 = -C_2$ we verify that $\nu\left(t, f(t)\right) = 0$ for all $t \geq 0$ in the case in which the boundary is

$$f(t) = -\frac{d_0}{2}t.$$

5.2. The quadratic boundary

Next we study a quadratic boundary. Taking $c_0, c_1 \neq 0$ in (35) we have that

$$\phi^{(2)}(x) = c_1 x\phi^{(0)}(x) + c_0\phi^{(0)}(x), \tag{44}$$

which is the Airy differential equation [17]. To solve (44) first consider the homogeneous Airy equation

$$\phi''(x) - x\phi(x) = 0. \tag{45}$$

Using Fourier transform we can find a solution $\phi(x)$ to (45) such that

$$\lim_{x\to+\infty}\phi(x) = 0.$$

It is given by

$$A_i(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xz+z^3/3)}dz. \tag{46}$$

Direct calculations show that a solution $\phi(x)$ to (44) is given by $A_i^{c_1}(x + \frac{c_0}{c_1})$, where

$$A_i^{c_1}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(xz+\frac{z^3}{3c_1})}dz = c_1^{1/3}A_i(c_1^{1/3}x). \tag{47}$$

In particular, the convolution of $A_i^{c_1}(x + \frac{c_0}{c_1})$ with the fundamental solution of the heat equation is

$$\nu(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\lambda x - \frac{\lambda^2 t}{2} + ic_0\frac{\lambda}{c_1} + i\frac{\lambda^3}{3c_1}\right)d\lambda \tag{48}$$

$$= \exp\left(\frac{c_1^2 t^3}{12} + \frac{xc_1 t}{2} + \frac{tc_0}{2}\right)A_i^{c_1}\left(x + \frac{c_0}{c_1} + \frac{c_1 t^2}{4}\right). \tag{49}$$

Hence, if we take $c_0 = c_1^{2/3} c_n$, where c_n is a root of $A_i(x)$ we obtain

$$\nu(t,x) = \exp\Big(\frac{c_1^2 t^3}{12} + \frac{x c_1 t}{2} + \frac{t c_1^{2/3} c_n}{2}\Big) A_i^{c_1}\Big(x + \frac{c_n}{c_1^{1/3}} + \frac{c_1 t^2}{4}\Big). \tag{50}$$

From (50) we note that $\nu(t, f(t)) = 0$ for all $t \geq 0$ in the case in which

$$f(t) = -\frac{c_1}{4} t^2.$$

We know that the zeros c_n of $A_i(x)$ are negative, and countable; see [17] p. 15.

5.3. The square root boundary

Next we analyze a square root boundary. To this end, from Theorem 4.3 we study the equation

$$\phi^{(2)}(x) = x d_1 \phi^{(1)}(x) + c_0 \phi^{(0)}(x),$$

whose solution has Fourier transform

$$\overline{\phi}(\lambda) = C_1 \lambda^{\frac{c_0 - d_1}{d_1}} e^{\frac{\lambda^2}{2 d_1}}.$$

In particular, if we let $s = -1/d_1 > 0$ and $z_0 = c_0/d_1$, then

$$\nu(t,x) = \frac{1}{\pi} 2^{\frac{z_0}{2} - 1} (s+t)^{-\frac{z_0+1}{2}} I,$$

where

$$\begin{aligned} I = \; & \Bigg[\sqrt{2} x \cos\left[\tfrac{z_0 \pi}{2}\right] \Gamma\left[\tfrac{1+z_0}{2}\right] \; {}_1F_1\left[\tfrac{1+z_0}{2}, \tfrac{3}{2}, -\tfrac{x^2}{2(s+t)}\right] \\ & + \sqrt{s+t}\,\Gamma\left[\tfrac{z_0}{2}\right] \; {}_1F_1\left[\tfrac{z_0}{2}, \tfrac{1}{2}, -\tfrac{x^2}{2(s+t)}\right] \sin\left[\tfrac{z_0 \pi}{2}\right] \Bigg], \end{aligned}$$

where ${}_1F_1$ is the confluent hypergeometric function (see [23] p. 503) and $\Gamma(\cdot)$ is the gamma function ([23] p. 253). To verify that with $f(t) = \sqrt{s+t}$ we obtain $\nu(t, f(t)) = 0$ for all $t \geq 0$, we use the previous expression to obtain

$$\nu(t, \sqrt{s+t}) = \frac{1}{\pi} 2^{\frac{z_0}{2} - 1} (s+t)^{-\frac{z_0}{2}} I$$

where

$$\begin{aligned} I = \; & \Bigg[\sqrt{2} \cos\left[\tfrac{z_0 \pi}{2}\right] \Gamma\left[\tfrac{1+z_0}{2}\right] \; {}_1F_1\left[\tfrac{1+z_0}{2}, \tfrac{3}{2}, -\tfrac{1}{2}\right] \\ & + \Gamma\left[\tfrac{z_0}{2}\right] \; {}_1F_1\left[\tfrac{z_0}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right] \sin\left[\tfrac{z_0 \pi}{2}\right] \Bigg]. \end{aligned}$$

We conclude by noticing that I is independent of s and t. Furthermore, by properties of the hypergeometric functions one can check that I has countably many roots as a function of c_0.

5.4. Remarks

1. In Theorem 4.3 the coefficient c_0 is independent of the boundaries. In Section 7 below we will use c_0 as an eigenvalue in the standard Sturm-Liouville theory to find densities of hitting times for Brownian motion.
2. We note that the coefficient of $\nu^{(2)}$ in equation (39) is given by

$$(1 - d_1 t - c_2 t^2).$$

In turn, d_0 and c_1 corresponded to the linear and quadratic boundaries, respectively. By analogy, we are tempted to study an ODE that leads to a solution of the heat equation involving a coefficient where t is of cubic order. We will do so in the next section.

6. Derivation of the cubic boundary

In this section we derive a function f which corresponds to a solution of the heat equation (2)-(3) with cubic absorbing boundary. We will make use of part 2 in Remark 5.4.

Theorem 6.1. *Suppose that the moving boundary f in (3) is $f(t) = -\frac{b}{8}t^3$. Furthermore, take $b \in \mathbb{R}\backslash\{0\}$ and let ϕ be a real-valued function that satisfies*

$$\phi'''(x) = bx^2\phi(x). \tag{51}$$

Then there exists a real-valued solution of (51), that convolved with the heat kernel yields a function ν that solves problem (2)-(3).

Before proving the theorem, we provide an example.

Example 6.2. *For $b = -1$, the function*

$$\phi(x) := \frac{x}{5^{3/5}} \cdot {}_0F_2\left[\Big\{\Big\}, \Big\{\frac{4}{5}, \frac{6}{5}\Big\}, -\frac{x^5}{125}\right],$$

defined in terms of the generalized hypergeometric function ${}_0F_2$ (see [24]), solves (51). Then the convolution of the heat kernel ω and ϕ,

$$\nu(t,y) = \int_{-\infty}^{\infty} \omega(t,x)\phi(y-x)dx,$$

is a solution to the problem (2)-(3) when $f(t) = t^3/8$. That is,

$$\nu(t, t^3/8) = 0 \qquad \forall t \geq 0.$$

The proof of Theorem 6.1 is in the spirit of the proof of Theorem 4.3. The only difference is that the function ϕ that links ν and the boundary f in problem (2)-(3) is now C^3 instead of C^2.

Proof of Theorem 6.1. If ϕ is a solution to (51) and $\nu(t,x)$ denotes the convolution of ϕ and the fundamental solution to the heat equation, then a direct application of Proposition 4.1 to the ODE (51) yields

$$\nu^{(3)}(t,x) = bt^2\nu^{(2)}(t,x) + 2btx\nu^{(1)}(t,x) + (bx^2 + bt)\nu(t,x). \tag{52}$$

Now, if f is such that $\nu(t,f(t))=0$, then from (32) we obtain

$$\nu^{(3)}(t,f(t)) = (-2bt^2 f'(t) + 2btf(t))\nu^{(1)}(t,f(t)). \tag{53}$$

Moreover, differentiating (52) with respect to x we have

$$\begin{aligned}\nu^{(4)}(t,x) &= bt^2\nu^{(3)}(t,x) + 2btx\nu^{(2)}(t,x)\\ &\quad + (2bt + bx^2 + bt)\nu^{(1)}(t,x) + 2bx\nu(t,x)\end{aligned}$$

and again, if $\nu(t,f(t))=0$, from this last expression and (53) it follows that

$$\nu^{(4)} = (-2b^2t^4 f'(t) + 2b^2t^3 f(t) - 4btf(t)f'(t) + 3bt + bf^2(t))\nu^{(1)}. \tag{54}$$

On the other hand, (33) reads

$$\nu^{(4)} = -4f''(t)\nu^{(1)} - 4f'(t)(f'(t)\nu^{(2)} + \nu^{(3)}).$$

From this latter equation, if $\nu(t,f(t))=0$, then (53) yields

$$\nu^{(4)} = (-4f'' + 8(f')^3 + 8bt^2(f')^2 - 8btff')\nu^{(1)}. \tag{55}$$

Thus equating (54) and (55) we obtain

$$\begin{aligned}&-4f''(t) + 2f'(t)[4(f'(t))^2 + 4b_2t^2f'(t) - 2b_2tf(t) + b_2^2t^4]\\ &\qquad\qquad - b_2f(t)[2b_2t^3 + f(t)] - 3b_2t = 0. \end{aligned}\tag{56}$$

Finally, to verify the statement of the theorem, let $f(t) = \delta t^3$, so $f'(t) = 3\delta t^2$ and $f''(t) = 6\delta t$. Then substitute the values of f, f' and f'' in (56) to obtain

$$\begin{aligned}&-24\delta t - 3b_2t + 2(3\delta t^2)(4\cdot 9\delta^2t^4 + 4b_2t^2\cdot 3\delta t^2 - 2b_2t\delta t^3 + b_2^2t^4)\\ &\quad -b_2\delta t^3(2b_2t^3 + \delta t^3) = 0,\end{aligned}$$

or equivalently

$$\begin{aligned}&-3t(8\delta + b_2) + 6\delta t^2(36\delta^2t^4 + 12b_2\delta t^4 - 2b_2\delta t^4 + b_2^2t^4)\\ &\quad -b_2\delta t^3(2b_2t^3 + \delta t^3) = 0.\end{aligned}$$

Factorizing in terms of t and t^6 we have

$$\begin{aligned}&-3t(8\delta + b_2)\\ &\quad +\delta t^6(216\delta^2 + 72b_2\delta - 12b_2\delta + 6b_2^2 - 2b_2^2 - b_2\delta) = 0.\end{aligned}$$

Thus, for the latter equality to hold for all $t \geq 0$ we should have

$$\delta = -b_2/8.$$

But this also yields

$$216b_2^2/64 - 59b_2^2/8 + 4b_2^2 = 0,$$

and thus the proof is complete. □

The next section is devoted to computing the hitting time up to a family of quadratic boundaries making use of (21) and (22), and Theorem 4.3.

7. Hitting time of a Brownian motion to a quadratic boundary

In this section we will compute explicitly the density of the hitting time of a standard Brownian motion up to a quadratic boundary $f(t) = a + \frac{k}{4}t^2$ for $a > 0, k > 0$. We will use the solution to the heat equation with quadratic moving boundary computed above, in (50). Recall that we are looking for a solution $\kappa(t,x)$ to the heat equation on $[0,s) \times (0,\infty)$ that satisfies the conditions (21) and (22) where $f(t) = \frac{k}{4}t^2$.

We will use the solution $\nu(t,x)$ to the heat equation with quadratic moving boundary computed above (see (50)),

$$\nu(t,x) = \exp\left\{\frac{k^2t^3}{12} + \frac{ktx}{2} + \frac{k^{2/3}c_n t}{2}\right\} A_i^k\left(x + \frac{c_n}{k^{1/3}} + \frac{kt^2}{4}\right) \tag{57}$$

where c_n is a zero of the Airy function $A_i(x)$. To fulfill the condition (21) we will use c_n as an eigenvalue in the Sturm-Liouville theory as was pointed out in Remark 5.4. To this end recall that the Airy function

$$A_i(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xz+z^3/3)}dz \tag{58}$$

has countably many zeros (see [17] p. 20) on the negative real axis.

Let

$$\mathcal{A} := A_i^{-1}(\{0\}) = \{c_n \in \mathbb{R}^- : n = 0,1,\cdots, A_i(c_n) = 0, c_{n+1} < c_n\} \tag{59}$$

be the set of zeros of $A_i(x)$. It is known ([17] p. 88) that

$$\int_0^\infty A_i^2(x+c_n)dx = A_i'^2(c_n). \tag{60}$$

Consider the regular Sturm-Liouville problem

$$\phi''(x) - kx\phi(x) = \lambda\phi(x), \quad \text{with} \quad \phi(0) = 0, \quad \lim_{x\to\infty}\phi(x) = 0, \tag{61}$$

for $x \in [0,\infty)$, defined on the ideal domain of C^2 functions that satisfy the boundaries conditions. From (61), letting $h(x) = \phi(x - \frac{\lambda}{k})$ we obtain

$$h''(x) - kxh(x) = 0, \quad \text{with} \quad h\Big(\frac{\lambda}{k}\Big) = 0, \quad \text{for} \quad x \in [0,\infty). \tag{62}$$

Hence, given that we know the solution to the ODE in (62) and its zeros, it follows that eigenvalues and eigenfunctions to the Sturm-Liouville problem are

$$\lambda = k^{2/3}c_n, \quad A_i^k\Big(x + \frac{c_n}{k^{1/3}}\Big), \tag{63}$$

respectively, where $c_n \in \mathcal{A}$ and $A_i^k(x) := k^{1/3}A_i(k^{1/3}x)$ was defined at (47).

From classical Sturm-Liouville theory (see [21], Chapter 3) it follows that the eigenvalues (63) form a discrete unbounded set and that the eigenfunctions in (63) form a complete orthogonal set in $L^2[0,\infty)$.

To compute the norm in $L^2[0,\infty)$ of $A_i^k(x + \frac{c_n}{k^{1/3}})$, note that

$$\int_0^\infty (A_i^k(x + \frac{c_n}{k^{1/3}}))^2 dx = k^{1/3}\int_0^\infty A_i^2(y+c_n)dy = k^{1/3}A_i'^2(c_n)$$

where the last integral follows from (60). From these facts we have that

$$\left\{\frac{A_i^k(x + \frac{c_n}{k^{1/3}})}{k^{1/6}|A_i'(c_n)|}\right\}_{n=0}^\infty \tag{64}$$

is a complete orthonormal set on $L^2[0,\infty)$.

The so-called closure representation to Dirac's delta in terms of a complete orthonormal family $\{\varphi_n\}$ in $L^2(\mathbb{R})$ is given by

$$\delta_0(x-t) = \sum_{n=0}^{\infty} \varphi_n(t)\varphi_n(x), \tag{65}$$

where this equality is in the distributional sense; see [25], p. 89.

The Fourier coefficients associated to (64) for the derivative of Dirac's delta are

$$\langle -\delta_0'(x), \frac{A_i^k(x - c_n/k^{1/3})}{k^{1/6}A_i'(c_n)}\rangle = (-1)^n k^{1/2}$$

Therefore, in the distributional sense we have that

$$-\frac{\delta_0'(x)}{2} = \sum_{n=0}^{\infty} k^{1/3}\frac{A_i^k(x + c_n/k^{1/3})}{2A_i'(c_n)}. \tag{66}$$

Taking convolution with respect to the heat kernel we obtain

$$\eta(t,x) = \sum_{n=0}^{\infty} \frac{k^{1/3}}{2A_i'(c_n)} e^{\frac{k^2t^3}{12} + \frac{ktx}{2} + \frac{k^{2/3}c_n t}{2}} A_i^k\left(x + \frac{c_n}{k^{1/3}} + \frac{kt^2}{4}\right). \tag{67}$$

Note that $\eta(t,x)$ is a solution to the heat equation that satisfies

$$\lim_{t\downarrow 0} \eta(t,x) = -\frac{1}{2}\delta_0'(x),$$

so, using η we can find a solution to (21). On the other hand, to satisfy the boundary condition (22) we use the following well known transformation

$$\kappa(t,x) = \exp\left(\mu x + \frac{1}{2}\mu^2 t\right)\eta(t, x+\mu t) \tag{68}$$

where μ is a constant given by $\mu = -f'(s) = -\frac{ks}{2}$. Note that κ is a solution to the heat equation and, furthermore,

$$\kappa(s-t, a + \int_t^s f'(u)du) = \sum_{n=0}^{\infty} \frac{k^{1/3}}{2A_i'(c_n)} I_n, \tag{69}$$

where I_n is given by

$$I_n = \exp\left(\frac{k^2}{12}(s-t)^3 - \frac{kta}{2} - \frac{k^2s^3}{8} + \frac{k^2t^3}{8} + \frac{k^2s^2t}{4} - \frac{k^2st^2}{4}\right) A_i^k\left(a + \frac{c_n}{k^{1/3}}\right). \tag{70}$$

Direct calculations show that κ satisfies the initial and boundary conditions (22)-(21). Hence, the density of the hitting time up to the quadratic boundary $a + \frac{k}{4}t^2$ is

$$\nu_f(s,a) = \kappa(s, a + \frac{k}{4}s^2) = \sum_{n=0}^{\infty} \frac{k^{1/3}}{2A_i'(c_n)} e^{-\frac{k^2s^3}{24} + \frac{k^{2/3}sc_n}{2}} A_i^k\left(a + \frac{c_n}{k^{1/3}}\right). \tag{71}$$

Remark 7.1. *This formula was computed in [12] (for $a = 1$ and $k = (2\kappa)^2$) with a different approach . In fact, the formula was obtained by means of the Laplace inversion formula and the residues theorem. In this general setting the formula can be seen in [26], Lemma 2.3.3. Although formula (71) has been computed in the literature ([12, 13]), there is no mention of the relationship between the boundary crossing problem up to a quadratic boundary and the heat equation.*

Example 7.2. *We have computed numerically the density of the hitting time of a one-dimensional Brownian motion to a quadratic boundary with $a = 1$, and $k = 4$ in Figure 1.*

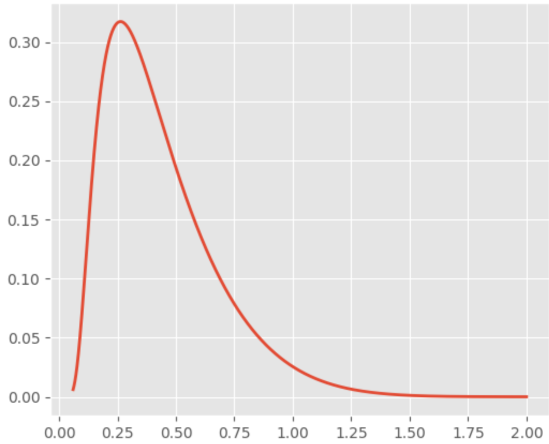

Figure 1: Density ν_f of the first hitting time of BM to a quadratic boundary with $a = 1$, and $k = 4$.

8. Concluding Remarks

In this work we give conditions under which the hitting time problem for Brownian motion is equivalent to solve a heat equation with non-standard initial and boundary conditions. This connection between boundary crossing problems and the heat equation opens a new way for studying hitting time densities for some smooth and convex boundaries. Motivated by hitting time problems we introduced a general framework to study the problem of the heat equation with absorbing moving boundaries for a family of boundaries. As an application of our results, we computed the density of the hitting time of a Brownian motion up to a quadratic boundary.

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