Financial Fragility Modelling and Applications
Section I: Basics[1]

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Course Outline

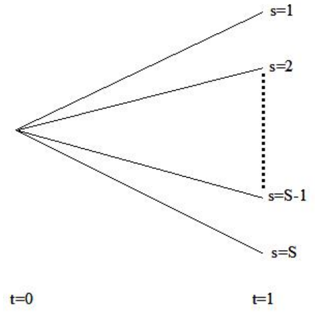
- <u>Section 1:</u>
  Basics
- <u>Section 2:</u>
  Default and expectations channels: Debt deflation and Minsky
- <u>Section 3:</u>
  Liquidity and Capital: Effects and Regulation
- <u>Section 4:</u>
  Integrated framework: Recent applications

Agenda

- Introduction
- Security markets
- Arbitrage pricing
- State prices
- Financial equilibrium

How to model uncertainty?

- We assume two time periods, today (t=0) and tomorrow (t=1)
- The present is certain
- We assume that $S = \{1, 2, ..., S-1, S\}$ different scenarios can happen at t=1
- We call these scenarios **states of the world**
- Each state is denoted by $s \in S$



Security Markets

Definitions

- A security j promises certain payments in each of these S states of the world-A security can be a stock, a bond, a derivative, any financial instrument
- Its price today is denoted by q_j
- Its payoff at state s is denoted by a_{js}
- $a_j = (a_{j1}, a_{j2}, ..., a_{j(S-1)}, a_{jS})'$ is the vector of payoff of security j for all states $s \in S$
- If we hold an amount θ_j of security j, then the vector $\theta_j^h a_j = (\theta_j^h a_{j1}, \theta_j^h a_{j2}, ..., \theta_j^h a_{j(S-1)}, \theta_j^h a_{jS})$ gives the payoffs of our holding for each state

Reason for holding securities: To transfer wealth from today to tomorrow

Portfolios of securities

Assume that there are J securities in the economy and each of one is defined by its payoff vector $a_j, j \in J$

- The payoff matrix of the economy is given by[2]

$$A = \begin{pmatrix} a_{11} & \cdots & a_{J1} \\ \vdots & \ddots & \vdots \\ a_{1S} & \cdots & a_{JS} \end{pmatrix}$$

- If we hold an amount θ_j of each security, then the total holdings are given by the vector $\theta^h = (\theta_1^h, \theta_2^h, ..., \theta_{j-1}^h, \theta_j^h)$
- The payoff we get at each state s is $\theta_1^h a_{1s} + \theta_2^h a_{2s} + ... + \theta_{j-1}^h a_{(J-1)s} + \theta_J^h a_{Js}$
- The payoff matrix for all states is

$$A \cdot \theta^h = \begin{pmatrix} a_{11} & \cdots & a_{J1} \\ \vdots & \ddots & \vdots \\ a_{1S} & \cdots & a_{JS} \end{pmatrix} \begin{pmatrix} \theta_1^h \\ \vdots \\ \theta_J^h \end{pmatrix} = \begin{pmatrix} \theta_1^h a_{11} + ... + \theta_J^h a_{J1} \\ \vdots \\ \theta_1^h a_{1S} + ... + \theta_J^h a_{JS} \end{pmatrix}$$

[2]Rows/Arrays represent states-Columns/Vectors represent assets

Asset Span

Intuitively the Asset Span is the "number" (a better word is dimension) of different payoff profiles one can construct via trade in the available assets

Formally the Asset Span is

$$M = \{z \in \Re^{S} : z = A \cdot \theta \text{ for some } \theta \in \Re^{J}\}$$

i.e. the set of different portfolio payoffs that we can achieve in the future via trade in the existing assets today

Definitions

Complete Markets

Markets are said to be complete iff **any newly** issued payoff profile can be constructed through buy and sell orders of the **existing** assets, which means that we can construct a portfolio h of the existing assets that will have the same payoff profile as the newly issued one

- Markets are complete if $M = \Re^S$
- This means that there are S securities with independent payoff profiles
- A security's payoff profile is independent when one cannot construct a portfolio of the remaining securities to replicate it
- If the payoff matrix X has rank S, then any conceivable security can be replicated and markets are complete

Definitions

Incomplete Markets

Not **every newly** introduced payoff can be replicated by a portfolio of the **existing** assets

- Markets are incomplete if $M \subset \Re^S$
- This means that there are less than S securities with independent payoff profiles
- Either there are less than S securities in the economy $(J < S)$ or one can construct a portfolio of the existing securities to replicate the payoff profile of some other existing securities
- If the payoff matrix X has rank less than S, then the markets are incomplete

Methods

Arbitrage Pricing

- No reference to preferences, endowments or production functions. Only the assumptions that agents prefer more to less
- Tells us how to price using the concept of the replicating portfolio
- State Pricing, Risk-neutral Valuation & Options Pricing

Equilibrium Pricing

- Analytical solution of a General Equilibrium maximization problem with specific reference to preferences, endowments and production functions
- Tells us how prices of traded securities are formed and how we should price newly issued ones
- **Lucas Model & Financial Equilibrium, GEI & CAPM**

Motivation

- In arbitrage pricing, prices are taken where given
- From the Equilibrium Condition $\Rightarrow$ No-Arbitrage Condition.
- From NA Condition $\Rightarrow$ Existence of Positive State Prices and Martingale Pricing (Fundamental Theorem of Finance).
- Question : Where do the asset prices come from? Welfare?

An Arrow-Debreu Economy

- The Arrow-Debreu model has two time periods ($t = 0, 1$) and S different states of the world ($s = 1, ..., S$), representing the different ways in which exogenous uncertainty might be resolved between periods 0 and 1
- The set of all possible outcomes is define by $S^* = \{0\} \cup S$
- We consider an environment with L commodities and H agents
- The key idea is that of a contingent commodity: a commodity whose delivery is conditional on the realized state of the world
- We assume an exchange economy in which agents are endowed with commodities both at t=0 and in every state at t=1, and choose to trade today the contingent commodities

Definition

A contingent commodity ls^* is a contract that promises the delivery of one unit of commodity $l \in L$ if state $s^* \in S^*$

An Arrow-Debreu Economy

- The endowments of agent h of the contingent commodity is given by the following vector:
$$e^h = \left(e^h_{10}, \ldots, e^h_{L0}, \ldots, e^h_{1S}, \ldots, e^h_{LS}\right) \in \mathbb{R}^{LS^*}$$
- His consumption plan can be characterized by the following vector:
$$x^h = \left(x^h_{10}, \ldots, x^h_{L0}, \ldots, x^h_{1S}, \ldots, x^h_{LS}\right) \in \mathbb{R}^{LS^*}$$
- The prices of the contingent commodities are:
$$\pi = (\pi_{10}, \ldots, \pi_{L0}, \ldots, \pi_{1S}, \ldots, \pi_{LS}) \in \mathbb{R}^{LS^*}$$
- Although deliveries are contingent on the state of the world, purchase happens today. Thus every agent faces one budget constraint:
$$B^h(\pi) = \left\{x \in \mathbb{R}^{LS^*} : \pi \cdot (x - e^h) \leq 0\right\}$$

An Arrow-Debreu Economy

Definition of Equilibrium

An **Arrow-Debreu equilibrium** is a $(\pi, (x^h)_{h \in H})$ such that $\pi \in \mathbb{R}^{LS^*}$ and $x^h \in \mathbb{R}^{LS^*}$ satisfying

1. $\sum_{h=1}^{H} x^h = \sum_{h=1}^{H} e^h$
2. $x^h \in B^h(\pi) \equiv \left\{ x^h \in \mathbb{R}^{LS^*} : \pi \cdot (x^h - e^h) \leq 0 \right\}$
3. $x \in B^h(\pi)$ implies $u^h(x) \leq u^h(x^h)$.

From the FOCs the price of contingent commodity $\{ls\}$ in terms of the price of (the numeraire) commodity $\{10\}$[3] is

$$\frac{\pi_{ls}}{\pi_{10}} = \frac{u'^h(x^h_{ls})}{u'^h(x^h_{10})}$$

[3] Commodity $\{10\}$, i.e. the first commodity at t=0, is chosen as the numeraire commodity

Asset Markets

- In the Arrow-Debreu model of general equilibrium, all commodities are traded at once, no matter when they are consumed, or under what state of nature
- All consumers face only one budget constraint
- The distinguishing feature of the asset markets model is that consumers face a multiplicity of budget constraints, at different times and under different states of nature
- To transfer wealth among budget constraints, consumers must hold assets
- Retrading at t=1 is not necessary for the Arrow-Debreu equilibrium
- Even if spot markets for all the L commodities were operating at t=1 and in each state, trade in these markets would not occur (prove this by contradiction)

Asset Markets

- In asset markets equilibrium retrading at t=1 is important and agents have to correctly anticipated today the price that will prevail tomorrow-Rational Expectations
- The Arrow-Debreu equilibrium is equivalent to an asset equilibrium with complete markets
- In the former there are $L \times S^*$ markets; in the latter there are $2L + S$ since only one state will be realized in tomorrow
- However, agents need to be more computationally capable in the latter

Financial Markets Equilibrium

- There is only one good in the economy
- An agent $h \in H$ consumes x_0^h at t=0 and $x_1^h = (c_{11}^h, .., c_{1S}^h)$ in every of the S states at t=1
- He obtains utility $u^h(x_0^h, x_1^h)$
- He is endowed with e_0^h today and $e_1^h = (e_{11}^h, ..., e_{1S}^h)$ in every state $s \in S$ tomorrow
- He can transfer wealth from today to tomorrow and across the S states via trade in J securities
- Thus he forms a portfolio $\theta^h = (\theta_1^h, .., \theta_J^h)$ of the J securities
- His objective is to maximize utility subject to his budget constraints
- His maximization problem is

$$\max_{x_0, x_1, \theta} u^h(x_0^h, x_1^h)$$

$$s.t.\ x_0^h \leq e_0^h - q\theta^h$$

$$x_1^h \leq e_1^h + A\theta^h$$

Solution

Constrained Optimization Problem

- Kuhn-Tucker methodology
- An interior solution for prices is

$$q = A\frac{\frac{\partial u^h}{\partial x_1^h}}{\frac{\partial u^h}{\partial x_0^h}}$$

which should be interpreted as

$$q_j = \sum_s a_{js}\frac{\frac{\partial u^h}{\partial x_1^h}}{\frac{\partial u^h}{\partial x_0^h}}$$

Market Clearing

- Goods market
  $\sum_h x_0^h \leq \sum_h e_0^h$ and $\sum_h x_{1s}^h \leq \sum_h e_{1s}^h$ for all states s
- Securities Market
  $\sum_h \theta_j^h = 0$

A solution to the above problem that every agent h maximizes his utility subject to his budget constraints and market clearing conditions hold is called a **Financial Equilibrium-FE**. Under very general hypothesis a FE exists. We should interpret observed prices as equilibrium prices

Complete Markets

- If markets are complete we can replace the A matrix with the identical $S \times S$ matrix and the vector of prices q with the vector of state prices π
- From the FOCs the state price for state s is $\pi_s = \frac{\frac{\partial u^h}{\partial x^h_{\mathbf{1}s}}}{\frac{\partial u^h}{\partial x^h_{\mathbf{0}}}}$

No arbitrage property

If utilities are strictly increasing then in a financial equilibrium there is no arbitrage

Intuition: With strictly increasing utility function the budget constraints are binding. If there was an arbitrage opportunity an agent could increase his utility without affecting his budget constraints in a negative way (remember the definition of arbitrage). Thus the budget constraints would not be binding anymore which is a contradiction. The price will adjust until equilibrium is reached

Properties

Redundant Securities

Redundant securities will never be traded in a Financial Equilibrium
They will be, however, priced via the arbitrage condition

Intuition: For any redundant security an agent can form a portfolio of the other securities to replicate its payoff. To optimize he does not need to take a position in a redundant security. When solving for a financial equilibrium we should not include redundant securities in the maximization problem

Properties

Sufficient condition for Pareto Optimality: The marginal rates of substitution (MRS) across states of the world are equal for all agents i.e.

$$MRS^A = MRS^B \Rightarrow \frac{\frac{\partial u^A}{\partial x^A_{\mathbf{1}s}}}{\frac{\partial u^A}{\partial x^A_{\mathbf{1}s'}}} = \frac{\frac{\partial u^B}{\partial x^B_{\mathbf{1}s}}}{\frac{\partial u^B}{\partial x^B_{\mathbf{1}s'}}}$$

Welfare

- The Arrow-Debreu equilibrium is Pareto Optimal-First Welfare Theorem
- When markets are complete the asset markets equilibrium is Pareto Optimal
- The reason is that state prices are unique, thus the MRSs of agents are equated
- When markets are incomplete FE is constrained optimal ($L = 1$)
- When markets are incomplete the asset markets equilibrium is constrained suboptimal ($L > 1$)

On Modeling Default

- *"Default is to macro-economics what sin is to theology: regrettable but central and essential".*

  Charles Goodhart

- Default is costly
  - To both, creditor and debtor (via punishment).
  - To market (negative externality worsened by adverse selection)
  - Assets may not be traded!

- When markets are incomplete
  - Provides extra insurance (spanning) opportunities to creditors.
  - Allows debtor to customize his own insurance opportunities.
  - Agents can be on both sides of the market (pooling).

On Modeling Default

- Default as a (General) Equilibrium phenomenon

  +incomplete markets

  +money

  =FINANCIAL FRAGILITY

- Regulation affects default, insurance (spanning), and financial trade

  $\Rightarrow$ Welfare depends on regulatory intervention.

On Modeling Default

- Default is a decision (variable) or bad luck
- Continuous/non-pecuniary penalties (Dubey, Geanakoplos and Shubik, 2005)

  or

  Cost reflected in margins (collateral)

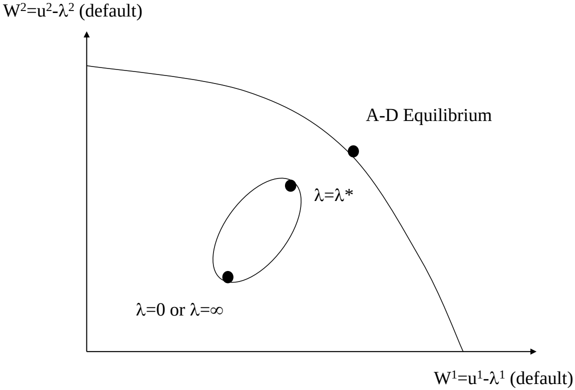
Background

- GENERAL EQUILIBRIUM THEORY has for the most part not made room for default
- In the Arrow-Debreu model of general equilibrium with complete contingent markets (GE), and likewise in the general equilibrium model with incomplete markets (GEI), agents keep all their promises by assumption
- More specifically, in the GE model, agents never promise to deliver more goods than they personally own. In the GEI model, the definition of equilibrium allows agents to promise more of some goods than they themselves have, provided they are sure to get the difference elsewhere
- Agents there too must honor their commitments, though no longer exclusively out of their own endowments. Each agent can keep his promises because other agents keep their promises to him
- When default in promises (assets) is allowed then the need to impose default penalties arise, since otherwise agents would default completely and essentially no asset would be traded

An intermediate level of default penalties can make everyone better-off

Shubik & Wilson, 1977 (Journal of Economics)

Dubey, Geanakoplos & Shubik, 2005 (Econometrica)



Phenomenon

Default changes spanning opportunities

$$\begin{pmatrix} 1 & 1-d_1 \\ 1 & 1-d_2 \\ 1 & 1-d_3 \end{pmatrix} \to \begin{pmatrix} 1 & 1-d_1' \\ 1 & 1-d_2' \\ 1 & 1-d_3' \end{pmatrix}$$

Default is endogenous, thus the asset span of defaultable assets will be endogenous as well. In general, the levels of default will be different $d_s \neq d_s'$

Economy

- Two agents, $h \in \{\alpha, \beta\}$
- Two states of the world, $s \in S = \{s_1, s_2\}$
- Logarithmic utility function-Consume both today and tomorrow
- One consumable good
- A "defaultable" asset that promises 1 in each state of the world
- A default penalty λ for both states and both agents
- Endowments $e^h = (e_0, e_1, e_2)$
  $e^\alpha = (1, 1, 0)$ and $e^\beta = (1, 0, 1)$
- Probabilities $\pi(s_1) = 1/2$ and $\pi(s_2) = 1/2$

Optimization Problem

$$\Pi = \ln x_0 + \pi_1 \ln x_1 + \pi_2 \ln x_2 - \lambda \sum_s \pi_s \max[\phi - D_s, 0]$$

subject to the following budget constraints:

$$x_0 + q(\theta - \phi) = 1 \quad (\mu_0)$$
$$x_1 + D_1 = \theta K_1 + 1 \quad (\mu_1)$$
$$x_2 + D_2 = \theta K_2 \quad (\mu_2)$$

where:

- Π is the payoff of the agents (utility less the cost of default)
- x_s^h is the consumption of agent h in state s, θ is the units of the asset bought
- ϕ is the units of the bond sold by the agent (and hence how much he promises to repay in the future)
- D_s is the amount actually delivered in state s
- The amount of default is given by $\phi - D_s$
- K_s is the delivery rate in state s. It is a macrovariable and is determined through market clearing

General Solution Method

Assume that there is only one Arrow security available for trade. Then, the variables that we have to solve for are 8:

$$\{K, x, \phi_1, \phi_2, \theta_1, \theta_2, D_1, D_2\}$$

Consumption is bad state is x. This means we need 8 equations:

1. On the verge condition for agent 1 selling the Arrow sec
2. On the verge condition for agent 2 selling the Arrow sec
3. Relationship between the marginal utility and disutility for 1 to see if he is strictly consientuous or not
   If $MU < MDU$ then full delivery $\Rightarrow \phi_1 = D_1$
   If $MU > MDU$ then complete default $\Rightarrow D_1 = 0$
   If $MU = MDU$ then partial default up to that level (an equation by its own)
4. Relationship between the marginal utility and disutility for 2 to see if he is strictly consientuous or not
   As above
5. Budget constraint for the state that the Arrow sec pays out for agent 1
6. Budget constraint for the state that the Arrow sec pays out for agent 2
7. $K = \frac{D_1 + D_2}{\phi_1 + \phi_2}$
8. $\phi_1 + \phi_2 = \theta_1 + \theta_2$

This solution method can be implemented for any number of securities, states and agents.

Collateral default

- A contract j is defined by a promise A_j made by the seller and the collateral C_j requires to back the promise
- A classic example might be the promise of \$100,000 backed by a house as collateral, which is called a mortgage
- The price $q_j = q(A_j, C_j)$ is the amount the buyer must pay the seller to obtain the contract
- contract j' making the same promise $A_{j'} = A_j$, but with a different collateral requirement, $C_{j'} \neq C_j$, is a different contract, and may sell for a different price $q_{j'} \neq q_j$
- If collateral is not observable then adverse selection issues arise. To abstract from them we assume collateral is observable, so one agent's collateral C_j is as good as another agent's collateral C_j

Any seller delivers in the future the minimum of the promise and the value of the collateral, since he is a utility maximizer. If at prevailing (equilibrium) prices in some state the value of the collateral is less than the the value of his obligation, then he would rationally choose to default

- It is straightforward that only durable (or storable) goods can serve as collateral
- Thus the collateral C_j will be taken to be a vector of durable, marketed goods
- Since C_j is marketed, in equilibrium one can always observe the value of the collateral $p(C_j$, as well as the price of the contract $q(A_j, C_j)$
- In equilibrium $q(A_j, C_j) \leq p(C_j)$, since by assumption the payoff from the contract will never exceed the value of the collateral (agents will choose to deliver in the future the lower of the two)

- Given the collateral requirement C_j for each contract j, the security it provided is each state s is

$$p_s \cdot f_s(C_j)$$

- $f_s : \mathbb{R}_+ \to \mathbb{R}_+$ is a depreciation function for each state s
- This collateral is owned by the borrower but may be confiscated by the lender (actually by the courts on behalf of the lender) if the borrower does not make his promised deliveries
- Since we have assumed that the borrower has nothing to lose but his collateral from walking away from his promise, it follows that the actual delivery by every agent h on asset j in state s will be:

$$D_{js}^h = \min\left[p_s \cdot A_{js}, p_s \cdot f_s(C_j)\right]$$

We can then easily defined the margin, m_j, on a contract (A_j, C_j) as:

$$m_j = \frac{p(C_j) - q(A_j, C_j)}{p(C_j)}$$

The margin is positive for three reasons:

1. The collateral may provide utility before the promises come due, boosting the price of the collateral above the price of the promise
2. There may be a mismatch between future collateral values and the promises, so that in some states the collateral is worth more than the promises
3. To the extend the mismatch is variable, risk averse lenders might prefer higher margins to higher interest rates (i.e., to lower prices q_j). If collateral was not observable then by doing so good borrowers are able to separate themselves from bad ones

The economy is defined by a vector

$$\mathbb{E} = \left(\left(u^h, e^h \right)_{h \in H}, (A_j, C_j)_{j \in J}, f \right)$$

The budget set of agent h can be written as:

$$B^h(p,q) = \{(x,\theta,\phi) \in \mathbb{R}^{L \times S^*} \times \mathbb{R}^J_+ \times \mathbb{R}^J_+ : p_{\mathbf{0}} \cdot (x^h_{\mathbf{0}} - e^h_{\mathbf{0}}) + q \cdot (\theta^h - \phi^h) \leq 0; \sum_j C_j \phi^h_j \leq x^{h,d}_{\mathbf{0}}$$

$$\text{and } p_s \cdot (x^h_s - e^h_s - f_s(x^{h,d}_{\mathbf{0}}) + \sum_j \phi^h_h \min\left[p_s A_{js}, p_s f_s(C_j)\right] \leq \sum_j \theta^h_h \min\left[p_s A_{js}, p_s f_s(C_j)\right]$$

$$\forall \text{ s} \in \text{S}\}$$

- Given the prices (p, q), each agents h decides what commodities to consume, x_0^h, where $x_0^{h,d}$ is the consumption of durable goods, what contracts to purchase, θ^h, and what contracts to sell ϕ^h
- Note that for every promise ϕ_j^h that he makes he must put up the corresponding collateral
- Thus, $\sum_j C_j \phi_j^h \leq x_0^{h,d}$
- After the state of nature is realized, the agent must again decide on his net purchases $x_s^h - e_s^h - f_s(x_0^{h,d})$. Recall that the goods x_0^d whose services are consumed at t=0 are durable, and thus still available for consumption at t=1, though they have depreciated
- These net expenditures on goods can be financed from the receipts from contracts j purchased at t=0, less the deliveries the agent makes on the contracts he sold at t=0

A **Collateral Equilibrium** is a $(p, q, (x^h, \theta^h, \phi^h)_{h \in H})$ satisfying

1. $\sum_{h=1}^{H} x_0^h = \sum_{h=1}^{H} e_0^h$
2. $\sum_{h=1}^{H} x_s^h = \sum_{h=1}^{H} e_s^h + f_s(e_0^{h,d})$
3. $\sum_{h=1}^{H} \theta^h = \sum_{h=1}^{H} \phi^h$
4. $\left(x^h, \theta^h, \phi^h) \in B^h(p, q)\right)$
5. $x \in B^h(q)$ implies $u^h(x) \leq u^h(x^h)$.

Back on track

Definition

*"An economic regime is said to be **financially unstable** if, inter alia, a 'number' (but not necessarily all) of households **and** banks defaults (i.e. liquidity crisis) without necessarily becoming bankrupt and the aggregate profitability of the banking sector decreases significantly (i.e. banking crisis)".*

Therefore, **Financial Instability** can be defined as any deviation from the optimal saving-investment plan of an economy that is due to imperfections in the financial sector. This is generally accompanied by a decrease of economic welfare.

Definition

We distinguish the following regimes:

1. **Liquidity crisis:** A situation in which there exists a 'number' of defaults (**not** necessarily bankruptcy and foreclosure) both in the household as well as the commercial banking sector.
2. **Banking crisis:** Together with default of a 'number' of commercial banks, it is required that the profitability of the banking sector as a whole decreases significantly.
3. **Recession:** In the context of an equilibrium model, a recession entails welfare losses by the aggregate household sector.

Comparison

	Concept	Price Stability	Financial Stability
a)	Measurement and Definition	Yes, subject to technical queries	Hardly, except by its absence
b)	Instrument for control	Yes, subject to lags	Varied, and difficult to adjust
c)	Accountable	Yes	Hardly
d)	Forecasting Structure	Central tendency of distribution	Tails of distribution
e)	Forecasting Procedure	Standard Forecasts	Simulations or Stress Tests
f)	Administrative Procedure	Simple	Difficult

Implied Model

This new trend theory implies a class of models defined by the inclusion of financial frictions in the General Equilibrium framework. The simplest case is the two periods model, where we can find most of the interesting features related to financial stability and monetary policy. This is extrated from Tsomocos (2003), JME.

Model

- Two periods, "s" states in second period (can be generalized).
- Households/Investors.
- Commercial Banks.
- (Endowed) commodities.
- OMO's on behalf of the Government/Central Bank.
- Money financed fiscal transfers.
- Fiscal policy.
- Regulation: (e.g. capital adequacy ratio prevailing in the commercial banking sector and bankruptcy penalties imposed upon the parties that abrogate their contractual obligations.)

Implied Model

Time structure